

# Basic theory of a class of linear functional differential equations with multiplication delay

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## Abstract

We study a class of linear functional differential equations with multiplication delay (also namely pantograph equation), such as  $y'(x) = y(\alpha x)$  where  $0 < \alpha < 1$ , and obtain the basic structures of solutions of initial value problem at the original point by using a kind of special functions namely exponent-like function, cosine-like function and sine-like function. We give some basic properties of these special functions, such as their addition formulae. By these special functions, we further get the existence, uniqueness and non-uniqueness of the initial value problems at a general point for the kind of linear pantograph equations. We also study the boundary value problem of the second order linear pantograph equations and obtain the results about the corresponding eigenvalues and eigenfunctions. These results show clearly and concretely that there are some serious differences between usual linear ordinary differential equations and linear pantograph equations.

**Key words:** pantograph equation; functional differential equation; multiplication delay; special function

## 1 Introduction

Usual functional differential equations such as  $y'(x) = y(x - x_0)$  also namely time-delay differential equation have been expansively studied from theories and applications[1]. Another kind of functional differential equations with the form  $y'(x) = y(\alpha x)$  where  $\alpha < 1$ , has many special properties so that it is difficult to deal with them. For example, we know that the energy is a conservative quantity for the second order vibration equation  $y''(t) = -y(t)$ , but for the equation  $y''(t) = -\alpha y(\alpha t)$ , the energy is not conserved since it is not invariant under the time translation. In fact, letting  $z(t) = y(t + T)$ , we have  $z''(t) = -\alpha y(\alpha t + \alpha T) \neq -\alpha z(\alpha t) = -\alpha y(\alpha t + T)$ . This implies that the original point

of time has a special meaning and can not be taken arbitrarily. If we notice that there is an original point of time for our universe which arises from the big bang, this kind of equations will become a possible mathematical tool to describe the corresponding physics.

Another possible application of this kind of equations is elastic-plastic mechanics[2] in which the strain of the material depends on subtraction delay type memory of the stress. Indeed, for instance, for an elastic-plastic spring, a memory effect can in general be represented in terms of a delay function  $h(t)$  such that the force at the time  $t$  depends on the displacement at the time  $t-h(t)$ . For simplicity, a routine way is to take a constant delay function  $h(t) = h_0$  which leads to a usual delay differential equation. But there exist some weaknesses for this choice. For example, the memory in  $t < h_0$  can not be considered. Moreover, the fixed  $h_0$  is only suitable for short period memory. For long time memory, a reasonable memory function should be a real function depending on a time variable  $t$ . The simplest choice is  $h(t) = \beta t$  with  $0 < \beta < 1$ , and hence  $t - h(t) = (1 - \beta)t = \alpha t$  with  $\alpha = 1 - \beta$ . This treatment leads naturally to a multiplication delay differential equation. Therefore, if replacing the subtraction delay by multiplication delay, the corresponding vibration equation will be  $y''(t) = -Ky(\alpha t)$  where  $0 < \alpha < 1$ .

This kind of functional differential equations (including the case of  $\alpha > 1$ ) have been studied for a long time. In 1940, Mahler[3] had introduced such type functional differential equations in number theory. In 1971, Fox et al[4] and Ockendon et al[5] proposed this kind of functional differential equations as the models to study some industrial problems. Kato and McLeod [6] studied the asymptotic properties of the solution of the equation  $y'(x) = ay(\lambda x) + by(x)$ . Carr and Dyson [7,8] studied the related problems on complex domain. In 1972, Morris, Feldstein and Bowen published an important paper[9] in which they obtained some very important results on the subject. In particular, they proved that the solution of the equation  $y'(x) = -y(\alpha x)$  with  $y(0) = 1$  has an infinity of positive zeroes, which means that at such zero points, the solutions of initial value problem will be not unique or do not exist. This kind of functional differential equations is also called the pantograph equations[10]. This is a convenient name, so we will often use it. Up to now, there have been a large number of papers to study this kind of equations in theory and applications(see, for example,[10-24,26-29]). Among these, Iserles[10,12] studied the generalized and nonlinear pantograph equations and gave some deep results, and Derfel and Iserles[13] dealt with the equation on the complex plane. Iserles and Liu[19] studied the integro-differential pantograph equation. Mallet-Paret and Nussbaum[20] discussed the analyticity and non-analyticity of solutions. As application, Van Brunt and Wake[22] studied a model in cell growth. Because there exist in general no solutions in terms of elementary functions and known special functions even for the simplest equation  $y'(x) = y(\alpha x)$ , Iserles and Liu[23] used the generalized hypergeometric functions to solve some integro-differential pantograph equations. Feldstein, Iserles and Levin[24] studied the embedding of this kind of equations into the infinite-dimensional ODE systems. Recently, Atiyah and Moore discussed a kind of functional differential equations

such as  $y'(x) = k(y(x - x_0) + y(x + x_0))$  in the study of some problems arising in fundamental physics [25]. This kind of delay differential equations can be solved by elementary functions. Atiyah and Moore pointed out that relativistic invariance implied that one must consider both advanced and retarded terms in the equations, and they named them as shifted equations and showed that the shifted Dirac equation had some novel properties and a tentative formulation of shifted Einstein-Maxwell equations naturally incorporates a small but nonzero cosmological constant. Following Atiyah and Moore[25], Kong and Zhang [26] also studied the pantograph type equations. Some new applications can be found in [27-29].

Although there are many theoretical results, the basic structures of the solutions to the linear pantograph equations need yet to further clarify. We want to study whether or not those well-known statements in the elementary theory of linear ordinary or partial differential equations can be generalize to the linear pantograph equations, such as the structures of the linear equations, variation of constant method for the non-homogenous equation  $y'(t) = py(\alpha t) + q(x)$ , and Fourier's method for boundary value problem of second order and so forth. In the present paper, we study the structures and representations of the solutions to this kind of linear pantograph equations. By introducing a kind of special functions namely exponent-like function, cosine-like function and sine-like function, we explicitly solve the initial value problems of these equations including the first order linear pantograph equation, the high order linear pantograph equation, the system of linear pantograph equations, and the boundary value problem of the second order pantograph equation. We give some properties of these special functions. By these special functions, the structures of the solutions of these linear pantograph equations are recovered and obtained clearly. In particular, we obtain the basic results on the existence, uniqueness and non-uniqueness of the solutions of initial value problem at a general point for linear pantograph equations. These results show some crucial differences between the usual linear differential equations and linear pantograph equations. For example, for some initial value problem at a general point, there does not exist solution or there are an infinity of solutions to these linear pantograph equations.

This paper is organized as follows. In section 2, as preliminary, the existence and uniqueness theorems for the initial value problem at the original point of linear pantograph equations are listed. Section 3 is a key section of the paper, in which three special functions namely exponential-like function  $E_\alpha(x)$ , sine-like function  $S_\alpha(x)$  and cosine-like function  $C_\alpha(x)$  are introduced and some properties are obtained by detailed analysis. These functions are the foundation of solving and studying the linear pantograph equations. In section 4, the non-homogenous pantograph equation  $y'(t) = py(\alpha t) + q(x)$  is solved by the power series method and a useful formula is given. This is a key result by which the structure of the solutions of the system of linear pantograph equations is obtained. In section 5, the solutions of the second order pantograph equation  $y''(x) + py'(\alpha x) + qy(\alpha^2 x) = 0$  are classified and its initial value problem at the original point is solved. In section 6, the system of the linear pantograph

equations is solved in details and the structure of the solutions is given. In section 7, an operator method is used to solve the high order linear pantograph equations and the corresponding formula of the solution is obtained. In section 8, we consider the initial value problem at a general point for linear pantograph equations, and obtain the results on the existence and uniqueness and non-uniqueness of solutions. In section 9, the boundary value problem of the second order linear pantograph equation is solved and the corresponding eigenvalues and eigenfunctions are given. In subsections 9.1 and 9.2, as applications of the second order boundary value problem, the heat-like pantograph equation and the wave-like pantograph equation are discussed, and their formal solutions represented by Fourier-like series in terms of the sine-like functions  $S_\alpha(x)$  are given. In last section, we give a short conclusion.

## 2 Preliminary: existence and uniqueness theorems at original point

Since the existence and uniqueness theorems are the foundations of the further studies, we give them as the preliminary in this section. For the following initial value problem of the first order pantograph equation

$$y'(x) = f(y(\alpha x)), \quad (1)$$

$$y(0) = y_0, \quad (2)$$

where  $0 < \alpha < 1$ , if  $f$  satisfies the Lipschitz condition, the existence and uniqueness of the local solution can be proven by the Banach fixed point theorem. In the paper, we only consider the linear equation, and hence we need not this result.

By the standard method in usual ODE theory[30], we can easily prove the following theorems (see the theorem 1 in [6]).

**Theorem 2.1.** There is a unique analytic solution for the initial value problem

$$\frac{dX(t)}{dt} = AX(\alpha t) + b(t), \quad (3)$$

$$X(0) = X_0, \quad (4)$$

where  $X(t) = (x_1(t), \dots, x_n(t))^T$  is a  $n$  dimensional column vector,  $b(t)$  is a known column vector function and  $A = (a_{ij})_{n \times n}$  is a constant matrix.

**Theorem 2.2.** If  $X_1(t), \dots, X_n(t)$  are  $n$  linear independent basic solutions of the homogenous equation of Eq.(3), and  $X^*(t)$  is a special solution of the non-homogenous equation, then the general solution of non-homogenous equation can be represented by linear combination of these basics solutions adding a special solution, i.e.,

$$X(t) = c_1 X_1(t) + \dots + c_n X_n(t) + X^*(t). \quad (5)$$

For example, for the first order linear pantograph equation

$$y'(x) = \beta y(\alpha x), \quad (6)$$

$$y(0) = y_0, \quad (7)$$

or the second equation

$$y''(x) = \beta y(\alpha x), \quad (8)$$

$$y(0) = y_0, y'(0) = v_0, \quad (9)$$

there is the unique analytic solution.

**Corollary 2.1.** If  $y_1(x)$  and  $y_2(x)$  are two linear independent solutions of the equation  $y''(x) = \beta y(\alpha x)$  with initial values  $y_1(0) = 1, y'(0) = 0$  and  $y_2(0) = 0, y'(0) = 1$  respectively, the general solution can be represented by

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad (10)$$

where  $c_1$  and  $c_2$  are arbitrary constants.

**Proof.** Firstly, by initial conditions, it is easy to see that these two solutions  $y_1$  and  $y_2$  are linear independent. Let  $y(x)$  be a solution with initial conditions  $y(0) = c_1$  and  $y'(0) = c_2$ , and take  $z(x) = c_1 y_1(x) + c_2 y_2(x)$ . Then  $y(x)$  and  $z(x)$  have the same initial values. From the unique theorem 2.1, it follows that  $y(x) = z(x)$ .

**Theorem 2.3.** For the  $n$ -th order linear pantograph equation

$$y^{(n)}(x) + p_{n-1} y^{(n-1)}(\alpha x) + p_{n-2} y^{(n-2)}(\alpha^2 x) + \cdots + p_1 y'(\alpha^{n-1} x) + p_0 y(\alpha^n x) = f(t), \quad (11)$$

$$y(0) = c_1, y'(0) = c_2, \dots, y^{(n-1)}(0) = c_{n-1}, \quad (12)$$

where  $f(t)$  is a known analytic function, there is a unique analytic solution, and the general solution of the Eq.(11) can be represented by linear combination of the basics solutions of homogenous equation adding a special solution of non-homogenous equation.

**Proof.** letting  $x_1(t) = y(t), x_2(t) = x'_1(\frac{t}{\alpha}), \dots, x_n(t) = x'_{n-1}(\frac{t}{\alpha})$ , the equation (11) is transformed to the first order linear equations system. By theorem 2.2, the proof is completed.

We can easily prove the following theorem.

**Theorem 2.4.** For the  $n$ -th order linear pantograph equation (11), if  $y_k(x)$  is the special solution of the equation (11) with  $f(t) = f_k(t)$ , then

$$y(x) = y_1(x) + \cdots + y_m(x)$$

is a special solution of Eq.(11) with  $f(t) = \sum_{k=1}^m f_k(t)$ .

### 3 Exponent-like function $E_\alpha(x)$ , cosine-like function $C_\alpha(x)$ and sine-like function $S_\alpha(x)$

In the section, we introduce three special functions which are the nontrivial generalizations of the usual exponential function, cosine function and sine function, and have some interesting properties such as infinite addition formulae. This section plays an important role in the paper since these special functions are the key mathematical tools and the foundation of all following studies.

We denote  $E_\alpha(x)$  the unique analytic solution of the following initial value problem

$$y'(x) = y(\alpha x), \quad (13)$$

$$y(0) = 1. \quad (14)$$

Then its power series expansion is given as follows

$$E_\alpha(x) = \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{x^n}{n!}. \quad (15)$$

Replacing  $x$  by  $ix$  yields

$$E_\alpha(ix) = C_\alpha(x) + iS_\alpha(x), \quad (16)$$

where  $i^2 = -1$  and

$$C_\alpha(x) = \sum_{n=0}^{+\infty} (-1)^n \alpha^{n(2n-1)} \frac{x^{2n}}{(2n)!}, \quad (17)$$

$$S_\alpha(x) = \sum_{n=0}^{+\infty} (-1)^n \alpha^{n(2n+1)} \frac{x^{2n+1}}{(2n+1)!}. \quad (18)$$

It is also easy to see that

$$C_\alpha(x) = \frac{E_\alpha(ix) + E_\alpha(-ix)}{2}, \quad (19)$$

and

$$S_\alpha(x) = \frac{E_\alpha(ix) - E_\alpha(-ix)}{2i}. \quad (20)$$

We call these three functions respectively the exponent-like function  $E_\alpha(x)$ , cosine-like function  $C_\alpha(x)$  and sine-like function  $S_\alpha(x)$ . When  $\alpha = 1$ , they all become the usual exponential function, cosine function and sine function. It is easy to show that these three special functions are analytic functions on whole complex plane, and  $C_\alpha(x)$  is an even function and  $S_\alpha(x)$  is an odd function, with  $C_\alpha(0) = 1$ ,  $C'_\alpha(0) = 0$ ,  $S_\alpha(0) = 0$  and  $S'_\alpha(0) = 1$ . In particular, we can easily prove the following important properties.

**Remark 3.1.** The form of the function  $E_\alpha(x)$  had been obtained in many papers such as [6,10,17]. Here, what I do is to give it a new name so that it

can be studied as an independent mathematical object and used as a convenient mathematical tool.

**Proposition 3.1.** For the first order derivative, we have

$$E'_\alpha(x) = E_\alpha(\alpha x), \quad (21)$$

$$C'_\alpha(x) = -S_\alpha(\alpha x), \quad (22)$$

$$S'_\alpha(x) = C_\alpha(\alpha x), \quad (23)$$

where prime means the derivative with respect to  $x$ .

**Proposition 3.2.** For the second order derivative, we have

$$C''_\alpha(x) = -\alpha C_\alpha(\alpha^2 x), \quad (24)$$

$$S''_\alpha(x) = -\alpha S_\alpha(\alpha^2 x), \quad (25)$$

where prime means the derivative with respect to  $x$ .

Furthermore, we have the following addition formulae.

**Proposition 3.3** (Addition formulae). The exponent-like function  $E_\alpha(x)$ , cosine-like function  $C_\alpha(x)$  and sine-like function  $S_\alpha(x)$  satisfy the addition formulae

$$E_\alpha(x+y) = \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{x^n}{n!} E_\alpha(\alpha^n y) = \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{y^n}{n!} E_\alpha(\alpha^n x); \quad (26)$$

$$C_\alpha(x+y) = \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n x^{2n}}{(2n)!} C_\alpha(\alpha^{2n} y) - \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n x^{2n+1}}{(2n+1)!} S_\alpha(\alpha^{2n+1} y); \quad (27)$$

$$S_\alpha(x+y) = \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n x^{2n}}{(2n)!} S_\alpha(\alpha^{2n} y) + \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n x^{2n+1}}{(2n+1)!} C_\alpha(\alpha^{2n+1} y). \quad (28)$$

**Remark 3.2.** When  $\alpha = 1$ , these formulae are just the usual addition formulae  $\exp(x+y) = \exp(x)\exp(y)$ ,  $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$  and  $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ .

**Proof.** Since the corresponding series are uniform convergent, the following summations can exchange orders. By direct computation, we have

$$\begin{aligned} E_\alpha(x+y) &= \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{1}{n!} (x+y)^n \\ &= \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k y^{n-k} \\ &= \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{x^n}{n!} \sum_{m=0}^{+\infty} \alpha^{\frac{m(m-1)}{2}} \frac{y^m}{m!} \alpha^{mn} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{x^n}{n!} E_\alpha(\alpha^n y) \\
&= \sum_{n=0}^{+\infty} \alpha^{\frac{n(n-1)}{2}} \frac{y^n}{n!} E_\alpha(\alpha^n x).
\end{aligned}$$

From the first formula (26), we have

$$C_\alpha(x+y) + C_\alpha(x-y) = 2 \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n x^{2n}}{(2n)!} C_\alpha(\alpha^{2n} y); \quad (29)$$

$$S_\alpha(x+y) + S_\alpha(x-y) = 2 \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n x^{2n+1}}{(2n+1)!} C_\alpha(\alpha^{2n+1} y); \quad (30)$$

$$C_\alpha(x+y) - C_\alpha(x-y) = -2 \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n x^{2n+1}}{(2n+1)!} S_\alpha(\alpha^{2n+1} y); \quad (31)$$

$$S_\alpha(x+y) - S_\alpha(x-y) = 2 \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n x^{2n}}{(2n)!} S_\alpha(\alpha^{2n} y). \quad (32)$$

By the above these formulae, we obtain the last two formulae (27) and (28). The proof is completed.

**Remark 3.3.** We define  $y = L_\alpha(x)$  to be the inverse function of  $x = E_\alpha(y)$  and call it the logarithm-like function. Therefore, we have

$$\frac{dy(x)}{dx} = \frac{1}{x(\alpha y)}.$$

By letting  $1+x = E_\alpha(y)$  and using  $\frac{d}{dx} = \frac{dy}{dx} \frac{d}{dy} = \frac{1}{E_\alpha(\alpha y)} \frac{d}{dy}$ , we derive the power series expansion of  $L_\alpha(1+x)$  as follows

$$L_\alpha(1+x) = x - \frac{\alpha}{2!} x^2 + \frac{\alpha^2(3-\alpha)}{3!} x^3 - \frac{\alpha^3(\alpha^3-6\alpha+11)}{4!} x^4 + \dots \quad (33)$$

If  $\alpha = 1$ , it is just the power series of the usual logarithm function  $\ln(1+x)$ .

On the zeros of cosine-like function  $C_\alpha(x)$  and sine-like function  $S_\alpha(x)$ , we have the following theorems.

**Theorem 3.1.** For  $0 < \alpha \leq 1$ , the cosine-like function  $C_\alpha(x)$  and sine-like function  $S_\alpha(x)$  have respectively an infinity of real zeroes.

**Proof.** We only need to prove the result of  $C_\alpha(x)$ . Since  $C_\alpha(x)$  satisfies the equation

$$y''(x) = -\alpha y(\alpha^2 x),$$

with  $y(0) = 1, y'(0) = 0$ , this means that the graph of  $C_\alpha(x)$  starts at the point  $(0, 1)$  and moves to the right with slope beginning at zero. By the equation itself, we know that  $y''(x) = -\alpha y(\alpha^2 x)$ , so when the curve is above the  $x$  axis,  $C_\alpha''(x)$  is negative that increases as the curve decreases, and hence the curve  $C_\alpha(x)$



bends down and crosses the  $x$  axis at some point  $\eta_0$ . Then at the point  $\frac{\eta_0}{\alpha^2}$  we have  $C''_\alpha(\frac{\eta_0}{\alpha^2}) = 0$  so the curve has an inflection point, and then the curve goes down continuously to the local lowest point  $\alpha\rho_1$  where  $\rho_1$  is the first positive zero point of  $S_\alpha(x)$  which exists by a similar discussion with  $C_\alpha(x)$ , and then the curve bends up and crosses  $x$  axis at some point  $\eta_1$ . This process will be continuous forever and gives an infinity of zeros. The proof is completed.

Next we give other properties of  $C_\alpha(x)$  and  $S_\alpha(x)$ . Denote  $\rho_0 = 0, \pm\rho_1, \dots, \pm\rho_n, \dots$  and  $\pm\eta_1, \dots, \pm\eta_n, \dots$  as the zeros of  $S_\alpha(x)$  and  $C_\alpha(x)$  respectively. It is easy to see that  $S_\alpha(\rho_n) = 0$  implies  $C'_\alpha(\frac{\rho_n}{\alpha}) = 0$  and vice versa.  $C_\alpha(x)$  is convex on the intervals  $[\eta_{2k}/\alpha, \eta_{2k+1}/\alpha]$ , and  $C_\alpha(x)$  is concave on the intervals  $[\eta_{2k+1}/\alpha, \eta_{2k+2}/\alpha]$ . The similar results hold for  $S_\alpha(x)$ . Furthermore, we have the following theorem.

**Theorem 3.2.** All real zeros of  $C_\alpha(x)$  and  $S_\alpha(x)$  are alternative each other, and their positive zeros satisfy

$$0 = \rho_0 < \alpha\eta_1 < \eta_1 < \alpha\rho_1 < \rho_1 < \alpha\eta_2 < \eta_2 < \alpha\rho_2 < \rho_2 < \dots. \quad (34)$$

**Proof.** Since  $S_\alpha(\rho_n) = S_\alpha(\rho_{n+1}) = 0$ , by Roll's theorem, there exists a point  $r_n \in (\rho_n, \rho_{n+1})$ , such that  $S'_\alpha(r_n) = 0$ , that is

$$C_\alpha(\alpha r_n) = S'_\alpha(r_n) = 0.$$

It follows that  $\alpha r_n$  is the  $(n-1)$ -th positive zero of  $C_\alpha(x)$ , that is,  $\eta_{n-1} = \alpha r_n$ , and then,

$$\alpha\rho_{n-1} < \eta_n < \alpha\rho_n.$$

Similarly, we have

$$\alpha\eta_n < \rho_n < \alpha\eta_{n+1}.$$

The proof is completed.

**Theorem 3.3** (The comparison theorem of the first positive zero). If  $y(x)$  is the solution of equation

$$y''(x) = -ky(\alpha^2 x),$$

with  $k > 0$  and initial conditions  $y(0) = 1, y'(0) = 0$  or  $y(0) = 0, y'(0) = 1$ , then the first positive zero  $x_0$  is a decreasing function as a function of  $k$  with  $\alpha$  fixed.

**Proof.** We only consider the case  $y(0) = 1, y'(0) = 0$ . Let  $y(x)$  and  $z(x)$  be respectively solutions of equations

$$y''(x) = -k_1 y(\alpha^2 x),$$

$$z''(x) = -k_2 z(\alpha^2 x),$$

with  $0 < k_1 < k_2$  and the same initial conditions. It is easy to see that  $z(x) = y(\sqrt{\frac{k_2}{k_1}}x)$  is the solution of the second equation with the initial conditions  $z(0) = 1, z'(0) = 0$ . We assume that  $x_0$  is the first positive zero of  $y(x)$ , then  $x_1 = \sqrt{\frac{k_1}{k_2}}x_0 < x_0$  is the first positive zero of  $z(x)$  since  $z(x_1) = y(x_0) = 0$ . The

conclusion in the case  $y(0) = 0, y'(0) = 1$  can be proven by similar method. The proof is completed.

**Proposition 3.4.**  $\rho_m$  and  $\eta_m$  satisfy the following identity,

$$\begin{aligned} & \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n (\rho_m - \eta_m)^{2n}}{(2n)!} S_\alpha(\alpha^{2n} \eta_m) \\ & + \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n (\rho_m - \eta_m)^{2n+1}}{(2n+1)!} C_\alpha(\alpha^{2n+1} \eta_m) = 0. \end{aligned} \quad (35)$$

**Proof.** Taking  $x = \rho_m - \eta_m$  and  $y = \eta_m$  in the formula (28) gives the result. The proof is completed.

This is a complicated relation between these zeros. Naturally, an interesting problem is to study whether there exists a simple relation between the  $n$ -th zeroes  $\rho_m$  and  $\eta_m$  of  $S_\alpha(x)$  and  $C_\alpha(x)$ . I leave it as an open problem.

On the integrals of  $C_\alpha(x)$  and  $S_\alpha(x)$ , we have the following results.

**Proposition 3.5.**

$$\int_{\frac{\rho_n}{\alpha}}^{\frac{\rho_{n+1}}{\alpha}} C_\alpha(\alpha^2 x) dx = \int_{\alpha \rho_n}^{\alpha \rho_{n+1}} C_\alpha(x) dx = 0, \quad (36)$$

$$\int_{\frac{\eta_n}{\alpha}}^{\frac{\eta_{n+1}}{\alpha}} S_\alpha(\alpha^2 x) dx = \int_{\alpha \eta_n}^{\alpha \eta_{n+1}} S_\alpha(x) dx = 0. \quad (37)$$

**Proof.** By  $C''_\alpha(x) = -\alpha C_\alpha(\alpha^2 x)$  and  $C'_\alpha(\frac{\rho_n}{\alpha}) = 0$ , we have

$$\int_{\frac{\rho_n}{\alpha}}^{\frac{\rho_{n+1}}{\alpha}} C_\alpha(\alpha^2 x) dx = -\frac{1}{\alpha} \int_{\frac{\rho_n}{\alpha}}^{\frac{\rho_{n+1}}{\alpha}} C''_\alpha(x) dx = -\frac{1}{\alpha} (C'_\alpha(\frac{\rho_{n+1}}{\alpha}) - C'_\alpha(\frac{\rho_n}{\alpha})) = 0.$$

Using variable transformation gives  $\int_{\alpha \rho_n}^{\alpha \rho_{n+1}} C_\alpha(x) dx = 0$ . Similarly, the last identity can be proven. The proof is completed.

**Remark 3.4.** On the computation of the first positive zeros of  $C_\alpha(x)$  and  $S_\alpha(x)$ , we now don't have a good method. Here, I propose an approximate approach to deal with it. Of course, it is not strict. Denote  $\eta_1(\alpha)$  as the first positive zero of  $C_\alpha(x)$  which can be considered as a function of  $\alpha$ . Here, what we want to do is to find its first order approximation. Since  $C_\alpha(x)$  is the solution of the equation  $y''(x) = -\alpha y(\alpha^2 x)$  with the initial conditions  $y(0) = 1$  and  $y'(0) = 0$ , we know that if  $\alpha = 0$ , the solution is  $y(x) = 1$  which has no zero, and if  $\alpha = 1$ , the solution is  $y(x) = \cos(x)$  whose first positive zero is  $\frac{\pi}{2}$ . Therefore, we can conclude that  $\eta_1(\alpha)$  is not an analytic function of  $\alpha$ , and has a singularity at  $\alpha = 0$ . Based on this observation, we assume that  $\eta_1(\alpha)$  has the following form

$$\eta_1(\alpha) = \alpha^\kappa \sum_{m=0}^{+\infty} x_m \alpha^m, \quad (38)$$

where  $\kappa$  is the index of singularity and  $x'_m$ s are the parameters undetermined. Under the first approximation, we can assume that  $x_0 + x_1 = \frac{\pi}{2}$ . Substituting  $\eta_1(\alpha)$  into the power series expansion of  $C_\alpha(x)$  yields

$$C_\alpha(\eta_1) = 1 - \frac{\alpha^{2\kappa+1}}{2}(x_0 + x_1\alpha + \cdots)^2 + \cdots = 0.$$

By taking the first order approximation, we have

$$1 - \frac{\alpha^{2\kappa+1}}{2}x_0^2 = 0,$$

which means

$$2\kappa + 1 = 0,$$

and then

$$1 - \frac{x_0^2}{2} = 0.$$

So we obtain  $\kappa = -\frac{1}{2}$  and  $x_0 = \sqrt{2}$ , and hence give the first order approximation  $\eta_1(\alpha) = \alpha^{-\frac{1}{2}}(\sqrt{2} + x_1\alpha)$ . By using  $x_1 = \frac{\pi}{2} - x_0$ , we have the following interesting formula

$$\eta_1(\alpha) \simeq \alpha^{-\frac{1}{2}}\{(1 - \alpha)\sqrt{2} + \alpha\frac{\pi}{2}\}. \quad (39)$$

We should notice that  $\sqrt{2}$  and  $\frac{\pi}{2}$  are respectively just the first positive zeros of the solution  $y(x) = 1 - \frac{x^2}{2}$  (for  $\alpha = 0$ ) and the solution  $y(x) = \cos(x)$  (for  $\alpha = 1$ ) of the equation  $y''(x) = -y(\alpha x)$  with  $y(0) = 1, y'(0) = 0$ .

By using the same method to deal with  $\frac{S_\alpha(x)}{x}$ , we have the first order approximation formula of the first positive zero  $\rho_1$  of  $S_\alpha(x)$ ,

$$\rho_1(\alpha) \simeq \alpha^{-\frac{3}{2}}\{(1 - \alpha)\sqrt{6} + \alpha\pi\}. \quad (40)$$

**Remark 3.5.** Next we consider the number theoretical properties of the zeroes of the sine-like function  $S_\alpha(x)$ . Notice that  $\rho_0 = 0, \pm\rho_1, \dots, \pm\rho_n, \dots$  are all real zeroes of the function  $S_\alpha(x)$ . We will have the following interesting formula

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \cdots + \frac{1}{\rho_n^2} + \cdots = \frac{\alpha^3}{6}. \quad (41)$$

In fact, formally, since the infinite product of  $S_\alpha(x)$  can be written as (at least formally)

$$\frac{S_\alpha(x)}{x} = (1 - \frac{x}{\rho_1})(1 + \frac{x}{\rho_1}) \cdots (1 - \frac{x}{\rho_n})(1 + \frac{x}{\rho_n}) \cdots,$$

and by the series expansion of  $S_\alpha(x)$ , we can obtain the formula (41).

It is easy to see that this formula gives the classical Euler's formula if we take  $\alpha = 1$ ,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Furthermore, we can also give the following formula

$$\frac{1}{\rho_1^4} + \frac{1}{\rho_2^4} + \cdots + \frac{1}{\rho_n^4} + \cdots = \frac{1}{36}\alpha^6 - \frac{1}{60}\alpha^{10}. \quad (42)$$

When  $\alpha = 1$ , the above formula gives the classical Euler's summation formula

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

Of course, by some efforts, we can give high order formulas. If more detailed information of zero points is known, we can get more exact formulas. I hope that these secrets will be recovered in future.

**Open problem 3.1.** Give the exact values of the zeros of the functions  $S_\alpha(x)$  and  $C_\alpha(x)$ .

## 4 Power series method for the first order linear non-homogenous pantograph equation

If non-homogenous term is a general smooth function, we do not know how to solve the first order linear pantograph equation since the method of the variation of constant can not be applied in this case. In this section, we use power series to solve the first order linear non-homogenous pantograph equation with analytic non-homogenous term. The power series method is simple but so powerful that we can use it to obtain the structure of the solutions for some important linear pantograph equations.

**Theorem 4.1.** Consider the following pantograph equation

$$y'(x) = \beta y(\alpha x) + q(x), y(0) = a_0, \quad (43)$$

where  $\beta$  is a constant, and  $q(x)$  is a analytic function with expansion

$$q(x) = \sum_{n=0}^{+\infty} q_n x^n.$$

Then its solution is given by

$$y(x) = a_0 E_\alpha(\beta x) + \sum_{n=1}^{+\infty} \left\{ \sum_{k=0}^{n-1} \frac{k! \beta^{n-k-1}}{\alpha^{\frac{k(k+1)}{2}}} q_k \right\} \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} x^n, \quad (44)$$

or another form, if the double summations in (44) can exchange each other,

$$y(x) = \left\{ a_0 + \sum_{k=0}^{+\infty} \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}} \right\} E_\alpha(\beta x) - \sum_{k=0}^{+\infty} \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}} \sum_{n=0}^k \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} \beta^n x^n. \quad (45)$$

**Proof.** Assuming that the power series expansion of  $y(x)$  is

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n,$$

and substituting it into the above equation and setting all coefficients of each term  $x^n$  to be zeros, we obtain the formulas of  $a_n$  as follows

$$a_{n+1} = \frac{\beta \alpha^n}{n+1} a_n + \frac{q_n}{n+1}.$$

Furthermore, we have (for  $n \geq 1$ )

$$a_n = \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} \{a_0 \beta^n + \sum_{k=0}^{n-1} \frac{k! \beta^{n-k-1}}{\alpha^{\frac{k(k+1)}{2}}} q_k\}.$$

So the solution can be represented by

$$\begin{aligned} y(x) &= \sum_{n=0}^{+\infty} a_0 \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} \beta^n x^n + \sum_{n=1}^{+\infty} \left\{ \sum_{k=0}^{n-1} \frac{k! \beta^{n-k-1}}{\alpha^{\frac{k(k+1)}{2}}} q_k \right\} \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} x^n \\ &= a_0 E_\alpha(\beta x) + \sum_{n=1}^{+\infty} \left\{ \sum_{k=0}^{n-1} \frac{k! \beta^{n-k-1}}{\alpha^{\frac{k(k+1)}{2}}} q_k \right\} \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} x^n, \end{aligned}$$

or by exchanging the summations order

$$y(x) = \{a_0 + \sum_{k=0}^{+\infty} \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}}\} E_\alpha(\beta x) - \sum_{k=0}^{+\infty} \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}} \sum_{n=0}^k \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} \beta^n x^n.$$

The proof is completed.

**Corollary 4.1.** If  $q(x)$  is a polynomial of  $n$  degree, the last summation will include only finite terms, that is,

$$y(x) = \{a_0 + \sum_{k=0}^n \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}}\} E_\alpha(\beta x) - \sum_{k=0}^n \frac{k! q_k}{\beta^{k+1} \alpha^{\frac{k(k+1)}{2}}} \sum_{n=0}^k \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} \beta^n x^n. \quad (46)$$

**Theorem 4.2.** If  $q(x) = q E_\alpha(\beta \alpha x)$ , the solution of Eq.(43) is given by

$$y(x) = c E_\alpha(\beta x) + q x E_\alpha(\alpha \beta x), \quad (47)$$

where  $c = a_0$ .

**Proof.** We can prove the result by using the theorem 4.1. In fact, according to the power series expansion of  $E_\alpha(\alpha \beta x)$ , for  $q(x) = q E_\alpha(\beta \alpha x)$ , we have

$$q(x) = \sum_{k=0}^{+\infty} q_k x^k = q \sum_{k=0}^{+\infty} \frac{\alpha^{\frac{k(k+1)}{2}}}{k!} \beta^k x^k$$

which gives

$$q_k = q \frac{\alpha^{\frac{k(k+1)}{2}}}{k!} \beta^k.$$

Substituting it into the solution (44) yields

$$\begin{aligned} y(x) &= a_0 E_\alpha(\beta x) + q \sum_{n=1}^{+\infty} \left\{ \sum_{k=0}^{n-1} \beta^{n-1} \right\} \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} x^n, \\ &= a_0 E_\alpha(\beta x) + q \sum_{n=1}^{+\infty} \beta^{n-1} \frac{\alpha^{\frac{n(n-1)}{2}}}{(n-1)!} x^n, \\ &= a_0 E_\alpha(\beta x) + qx \sum_{n=0}^{+\infty} \beta^n \frac{\alpha^{\frac{n(n+1)}{2}}}{n!} x^n, \\ &= a_0 E_\alpha(\beta x) + qx \sum_{n=0}^{+\infty} \beta^n \alpha^n \frac{\alpha^{\frac{n(n-1)}{2}}}{n!} x^n, \\ &= a_0 E_\alpha(\beta x) + qx E_\alpha(\alpha \beta x). \end{aligned}$$

The proof is completed.

**Remark 4.1.** Here we can give another derivation by undetermined coefficients method. Assume the solution has the form

$$y(x) = c E_\alpha(\gamma_1 x) + B x E_\alpha(\gamma_2 x),$$

where  $c, B, \gamma_1$  and  $\gamma_2$  are unknown parameters. Substituting it into equation and setting the coefficients of  $E_\alpha(\alpha \gamma_1 x)$  and  $E_\alpha(\alpha \gamma_2 x)$  to be zeros yields that  $c$  is an arbitrary constant and  $\gamma_1 = \beta, \gamma_2 = \alpha \beta$ , and then  $B = q$ .

**Theorem 4.3.** If  $q(x) = qx^k E_\alpha(\beta \alpha^{k+1} x)$  where  $k \neq -1$ , then the solution of Eq.(43) is given by

$$y(x) = c E_\alpha(\beta x) + \frac{q}{k+1} x^{k+1} E_\alpha(\alpha^{k+1} \beta x),$$

where  $c = a_0$ .

**Proof.** Assume that a special solution of the Eq.(43) is

$$y^*(x) = A x^h E_\alpha(\gamma x).$$

Then, we have

$$\frac{d}{dx} y^*(x) = A h x^{h-1} E_\alpha(\gamma x) + A x^h \gamma E_\alpha(\alpha \gamma x),$$

and

$$y^*(\alpha x) = A \alpha^h x^h E_\alpha(\alpha \gamma x).$$

Therefore, we get

$$A h x^{h-1} E_\alpha(\gamma x) + A x^h \gamma E_\alpha(\alpha \gamma x) = \beta A \alpha^h x^h E_\alpha(\alpha \gamma x) + q x^k E_\alpha(\beta \alpha^{k+1} x),$$

and hence we have

$$\gamma = \beta\alpha^h, qx^k = hAx^{h-1}, \gamma = \beta\alpha^{k+1},$$

from which it follows that

$$h = k + 1, \gamma = \beta\alpha^{k+1}, A = \frac{q}{k+1}.$$

Therefore, the special solution is

$$y^*(x) = \frac{q}{k+1}x^{k+1}E_\alpha(\alpha^{k+1}\beta x),$$

from which we get the conclusion. The proof is completed.

**Remark 4.2.** In some special cases, for the first order variable coefficient equations, the exact solutions can be obtained. For example, assuming that the solution of equation  $y'(x) = p(x)y(\alpha x)$  is  $y(x) = E_\beta(\phi(x))$ , then we have

$$y'(x) = p(x)E_\beta(\phi(\alpha x)) = \phi'(x)E_\beta(\beta\phi(x)),$$

which gives

$$\beta = \alpha^\gamma, \phi(x) = Ax^\gamma, p(x) = \gamma Ax^{\gamma-1},$$

that is,  $y(x) = y(0)E_{\alpha^\gamma}(Ax^\gamma)$  is the solution of equation  $y'(x) = \gamma Ax^{\gamma-1}y(\alpha x)$ .

## 5 The initial value problem of the second order linear pantograph equation

For the second order linear pantograph equation without the first order term

$$y''(x) = Ay(\gamma x), \tag{48}$$

$$y(0) = c_1, y'(0) = c_2, \tag{49}$$

where  $0 < \gamma < 1$ , we can solve it by the special function  $E_\alpha(x)$ . In fact, taking  $y(x) = E_\alpha(\beta x)$  and using

$$y''(x) = \alpha\beta^2y(\alpha^2x),$$

gives

$$\alpha\beta^2 = A,$$

$$\alpha^2 = \gamma.$$

Solving them yields  $\alpha = \sqrt{\gamma}$  and  $\beta = \pm\sqrt{\frac{A}{\sqrt{\gamma}}}$ . Therefore, we obtain two basic solutions

$$y_1(x) = E_{\sqrt{\gamma}}(\sqrt{\frac{A}{\sqrt{\gamma}}}x),$$

$$y_2(x) = E_{\sqrt{\gamma}}(-\sqrt{\frac{A}{\sqrt{\gamma}}}x),$$

and hence the general solution can be represented by

$$y(x) = a_1 E_{\sqrt{\gamma}}(\sqrt{\frac{A}{\sqrt{\gamma}}}x) + a_2 E_{\sqrt{\gamma}}(-\sqrt{\frac{A}{\sqrt{\gamma}}}x), \quad (50)$$

where  $a_1 = \frac{1}{2}(c_1 + c_2\sqrt{\frac{\sqrt{\gamma}}{A}})$  and  $a_2 = \frac{1}{2}(c_1 - c_2\sqrt{\frac{\sqrt{\gamma}}{A}})$ .

We now consider the general case and give the following results.

**Theorem 5.1.** For the second order linear pantograph equation with the first order term,

$$y''(x) + py'(\alpha x) + qy(\alpha^2 x) = 0, \quad (51)$$

$$y(0) = c_1, y'(0) = c_2, \quad (52)$$

where  $y'(\alpha x) = y'(t)|_{t=\alpha x}$ . There are the following two cases:

(i). If  $\Delta = p^2 - 4\alpha q \neq 0$ , the solution is given by

$$y(x) = a_1 E_{\alpha}(\beta_1 x) + a_2 E_{\alpha}(\beta_2 x), \quad (53)$$

where  $\beta_{1,2} = \frac{-p \pm \sqrt{\Delta}}{2\alpha}$ ,  $a_1 = \frac{c_1 \beta_2 - c_2}{\beta_2 - \beta_1}$  and  $a_2 = \frac{c_1 \beta_1 - c_2}{\beta_1 - \beta_2}$ .

(ii). If  $\Delta = p^2 - 4\alpha q = 0$ , the solution is given by

$$y(x) = a_1 E_{\alpha}(\beta x) + a_2 x E_{\alpha}(\alpha \beta x), \quad (54)$$

where  $\beta = -\frac{p}{2\alpha}$ ,  $a_1 = c_1$  and  $a_2 = c_2 - c_1 \beta$ .

**Proof.** Taking  $y(x) = E_{\alpha}(\beta x)$  and substituting it into the above equation and eliminating  $E_{\alpha}(\beta \alpha^2 x)$  gives the characteristic equation

$$\alpha \beta^2 + p\beta + q = 0. \quad (55)$$

Solving it yields

$$\beta_{1,2} = \frac{-p \pm \sqrt{\Delta}}{2\alpha},$$

where  $\Delta = p^2 - 4\alpha q$ . Therefore, if  $\Delta \neq 0$ , we obtain two basic solutions

$$y_1(x) = E_{\alpha}(\beta_1 x),$$

$$y_2(x) = E_{\alpha}(\beta_2 x),$$

and then the general solution can be represented by

$$y(x) = a_1 E_{\alpha}(\beta_1 x) + a_2 E_{\alpha}(\beta_2 x),$$

where the coefficients  $a_1$  and  $a_2$  can be determined by initial conditions.

If  $\Delta = 0$ , we can give a basic solution, that is,

$$y_1(x) = E_{\alpha}(\beta x).$$



In order to obtain another basic solution, since the usual method of variation of constant is not suitable, we must find other method. In section 7, we will use a kind of operator technics to deal with this problem and give the details of the corresponding theory. Here, we only verify that another basic solution is

$$y_2(x) = xE_\alpha(\alpha\beta x).$$

In fact, by inserting it into equation, it is easy to see that it is just a solution. Therefore, in this case, the general solution (54) is given. The proof is completed.

Next we consider the second order linear pantograph equations with some non-homogenous terms and give the following result.

**Theorem 5.2.** For the non-homogenous equation

$$y''(x) + py'(\alpha x) + qy(\alpha^2 x) = \sum_{k=1}^m A_k E_\alpha(r_k x),$$

where  $A_k$  and  $r_k$  are known constants, a special solution is given by

$$y^*(x) = \sum_{k=1}^m \frac{A_k \alpha^3}{r_k^2 + pr_k \alpha + q\alpha^3} E_\alpha\left(\frac{r_k}{\alpha^2} x\right), \quad (56)$$

where we require  $r_k^2 + pr_k \alpha + q\alpha^3 \neq 0$ . Further, the general solution is the summation of the general solution of the homogenous equation and the special solution.

**Proof.** Assuming that  $y_k^*(x) = B_k E_\alpha(s_k x)$  is the special solutions of the equation  $y''(x) + py'(\alpha x) + qy(\alpha^2 x) = A_k E_\alpha(r_k x)$  and substituting it into the equation yields

$$s_k = \frac{r_k}{\alpha^2}, B_k = \frac{A_k \alpha^3}{r_k^2 + pr_k \alpha + q\alpha^3}.$$

Therefore, from the theorem 2.4, we get the special solution (56). The proof is completed.

It is well-known that there is a simple conservation law for the usual vibration equation ( $\alpha = 1$ ) namely energy conservation law according to the invariance of time translation. But we do not know whether there exists a similar simple conservation law for the case of  $0 < \alpha < 1$ . A complicated conservation law can be given by using the addition formulas of  $S_\alpha(x)$  and  $C_\alpha(x)$ .

**Theorem 5.3.** For the second order pantograph equation

$$q''(t) = -\alpha q(\alpha^2 t), \quad (57)$$

$$q(0) = q_0, q'(0) = v_0, \quad (58)$$

if denote  $p(t) = q'(\frac{t}{\alpha})$ , then there is a conservation law as follows

$$I(t) = \frac{1}{q_0^2 + v_0^2} \sum_{n=0}^{+\infty} \frac{(-1)^n t^n}{n!} (q_0 q^{(n)}(t) + v_0 p^{(n)}(t)) = 1. \quad (59)$$

**Proof.** In the addition formula

$$C_\alpha(x+y) = \sum_{n=0}^{+\infty} \alpha^{n(2n-1)} \frac{(-1)^n x^{2n}}{(2n)!} C_\alpha(\alpha^{2n} y) - \sum_{n=0}^{+\infty} \alpha^{n(2n+1)} \frac{(-1)^n x^{2n+1}}{(2n+1)!} S_\alpha(\alpha^{2n+1} y),$$

we take  $y = -x = t$ , and notice that  $C_\alpha^{(2n)}(y) = (-1)^n \alpha^{n(2n-1)} C_\alpha(\alpha^{2n} y)$  and  $C_\alpha^{(2n+1)}(y) = (-1)^{n+1} \alpha^{n(2n+1)} S_\alpha(\alpha^{2n+1} y)$ , then we have

$$1 = C_\alpha(t-t) = \sum_{n=0}^{+\infty} \frac{(-1)^n t^n}{n!} C_\alpha^{(n)}(t). \quad (60)$$

Since the solution of the initial value problem is

$$q(t) = q_0 C_\alpha(t) + v_0 S_\alpha(t),$$

we have

$$C_\alpha(t) = \frac{q_0 q(t) + v_0 q'(\frac{t}{\alpha})}{q_0^2 + v_0^2} = \frac{q_0 q(t) + v_0 p'(t)}{q_0^2 + v_0^2}.$$

Then by substituting it into Eq.(60) gives the result. The proof is completed.

**Remark 5.1.** For the general second order equation  $y''(x) + py'(\alpha x) + qy(\beta x) = 0$  with  $\beta \neq \alpha^2$ , we need introduce new special functions to solve it.

**Remark 5.2.** We almost don't consider the real solutions when some eigenvalues are perhaps not real numbers. By means of the relations between  $E_\alpha(x)$  and  $S_\alpha(x)$  and  $C_\alpha(x)$ , we can easily obtain the corresponding results. In the paper, we omit them for simplicity.

**Open problem 5.1.** Give a variational principle for the second order vibration equation  $y''(x) = -ky(\alpha x)$ .

## 6 The system of the linear pantograph equations

**Theorem 6.1.** For the system of linear pantograph equations

$$x'_n(t) = \beta x_n(\alpha t) + x_{n+1}(\alpha t), n = 1, \dots, m, \quad (61)$$

where  $x_{m+1}(t) = 0$ , its solutions of initial value problem at the original point  $t = 0$  are given by

$$x_{m-k}(t) = \sum_{j=0}^k \frac{x_{m-k+j}(0)}{j!} \alpha^{\frac{j(j-1)}{2}} t^j E_\alpha(\alpha^j \beta t), \quad (62)$$

where  $k = 0, 1, \dots, m-1$ .

**Proof.** Firstly, we prove the theorem for  $k = 0$ . The corresponding equations become

$$x'_m(t) = \beta x_m(\alpha t),$$

whose solution is given by

$$x_m(t) = x_m(0)E_\alpha(\beta t).$$

Further, we consider

$$x'_{m-1}(t) = \beta x_{m-1}(\alpha t) + x_m(\alpha t),$$

that is

$$x'_{m-1}(t) = \beta x_{m-1}(\alpha t) + x_m(0)E_\alpha(\beta t).$$

By the theorem 4.2, we have

$$x_{m-1}(t) = x_{m-1}(0)E_\alpha(\beta t) + x_m(0)tE_\alpha(\alpha\beta t).$$

In general, we assume that the formula (62) holds for  $k$ , then we prove it holds for  $k+1$ . In fact, we have

$$x'_{m-k-1}(t) = \beta x_{m-k-1}(\alpha t) + x_{m-k}(\alpha t),$$

that is,

$$x'_{m-k-1}(t) = \beta x_{m-k-1}(\alpha t) + \sum_{j=0}^k \frac{x_{m-k+j}(0)}{j!} \alpha^{\frac{j(j+1)}{2}} t^j E_\alpha(\alpha^{j+1}\beta t).$$

From the theorems 2.4 and 4.3, we have

$$\begin{aligned} x_{m-k-1}(t) &= x_{m-k-1}(0)E_\alpha(\beta t) + \sum_{j=0}^k \frac{x_{m-k+j}(0)}{j!} \alpha^{\frac{j(j+1)}{2}} \frac{1}{j+1} t^{j+1} E_\alpha(\alpha^{j+1}\beta t) \\ &= x_{m-k-1}(0)E_\alpha(\beta t) + \sum_{j=0}^k \frac{x_{m-k+j}(0)}{(j+1)!} \alpha^{\frac{j(j+1)}{2}} t^{j+1} E_\alpha(\alpha^{j+1}\beta t) \\ &= \sum_{j=0}^{k+1} \frac{x_{m-k+j-1}(0)}{j!} \alpha^{\frac{j(j-1)}{2}} t^j E_\alpha(\alpha^j\beta t). \end{aligned}$$

By the mathematical induction method, the proof is completed.

According to the theorem 6.1, we have the following theorem.

**Theorem 6.2.** Consider the system of linear pantograph equations

$$\frac{dY(t)}{dt} = AY(\alpha t), \quad (63)$$

with initial condition

$$Y(0) = Y_0, \quad (64)$$

where  $y(t) = (y_1(t), \dots, y_n(t))^T$  is a vector function and  $A = (a_{ij})_{n \times n}$  is a constant matrix.

**Case (i).** If  $A$  has  $n$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , then there is an invertible matrix  $P$  such that  $P^{-1}AP$  is diagonal, i.e.,

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

and hence the equations become

$$x'_j(t) = \lambda_j x_j(\alpha t), j = 1, \dots, n,$$

whose solutions are given by

$$x_j(t) = x_j(0)E_\alpha(\lambda_j t), j = 1, \dots, n. \quad (65)$$

Respectively, we have the solutions  $Y(t) = PX(t)$ , and the general solution of the homogenous equation is the linear combination of these solutions.

**Case (ii).** If  $A$  has  $m$  distinct eigenvalues are  $\lambda_1, \dots, \lambda_r$  with multiplies  $n_1, \dots, n_r$ , and corresponding elementary factors are  $(\lambda - \lambda_1)^{k_{11}}, \dots, (\lambda - \lambda_1)^{k_{1m_1}}, \dots, (\lambda - \lambda_r)^{k_{r1}}, \dots, (\lambda - \lambda_r)^{k_{rm_r}}$  where  $k_{j1} + \dots + k_{jm_j} = n_j$  for  $j = 1, \dots, r$ , then there is an invertible matrix  $P$  such that  $P^{-1}AP$  is a Jordan form matrix, that is,

$$P^{-1}AP = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{m_1+\dots+m_r} \end{pmatrix}.$$

where  $J'_k$ s are the corresponding Jordan blocks. For every Jordan block, the solutions  $X$  can be given by formula (62). Then we have solutions  $Y = PX$ , and the general solution of the homogenous equation is the linear combination of these solutions.

## 7 Operator method

We use an operator method to deal with linear pantograph equations. Two basic operators are derivative operator  $D$  and the scale operator  $T_\alpha$  defined by

$$Dy(x) = \frac{dy}{dx},$$

$$T_\alpha y(x) = y(\alpha x).$$

Two basic properties of these two operators are

$$DT_\alpha = \alpha T_\alpha D,$$

$$T_{\alpha_1} T_{\alpha_2} = T_{\alpha_1 \alpha_2}.$$

**Theorem 7.1.** For a special first order equation

$$Dy(x) = pT_\alpha y(x) + qE_\alpha(\gamma x), \quad (66)$$

the general solution is given by

$$y(x) = cE_\alpha(px) + \frac{q\alpha}{\gamma - p\alpha} E_\alpha\left(\frac{\gamma}{\alpha}x\right), \quad (67)$$

where  $\gamma \neq p\alpha$  and  $c$  is an arbitrary constant which can be determined by  $y(0)$ .

**Proof.** Formally, we have

$$y(x) = q(D - pT_\alpha)^{-1}E_\alpha(\gamma x).$$

To compute the right side of the above equation, we use the following formula

$$(D - pT_\alpha)E_\alpha(\beta x) = (\beta - p)E_\alpha(\alpha\beta x).$$

So, taking  $\beta = \frac{\gamma}{\alpha}$  gives

$$(D - pT_\alpha)^{-1}E_\alpha(\gamma x) = \frac{1}{\frac{\gamma}{\alpha} - p} E_\alpha\left(\frac{\gamma}{\alpha}x\right).$$

Therefore, a special solution is given by

$$y^*(x) = \frac{q\alpha}{\gamma - p\alpha} E_\alpha\left(\frac{\gamma}{\alpha}x\right),$$

and the general solution is

$$y(x) = cE_\alpha(px) + \frac{q\alpha}{\gamma - p\alpha} E_\alpha\left(\frac{\gamma}{\alpha}x\right),$$

where  $c$  is an arbitrary constant. The proof is completed.

**Remark 7.1.** For  $\gamma = p\alpha$ , by theorem 4.2, the solution is given by

$$y(x) = cE_\alpha(px) + qx E_\alpha(p\alpha x).$$

Now we consider the general second order linear homogenous pantograph equation

$$y''(x) + py'(\alpha x) + qy(\alpha^2 x) = 0, \quad (68)$$

whose operator form is

$$(D^2 + pT_\alpha D + qT_\alpha^2)y(x) = 0.$$

In general, we have the operator decomposition

$$D^2 + pT_\alpha D + qT_\alpha^2 = (D - \lambda_1 T_\alpha)(D - \lambda_2 T_\alpha),$$

where the parameters  $\lambda_1$  and  $\lambda_2$  satisfy

$$\lambda_1 + \lambda_2 \alpha = -p, \lambda_1 \lambda_2 = q.$$

**Remark 7.2.** If we take  $\lambda_1 = \alpha\beta_1$  and  $\lambda_2 = \beta_2$ , then we will give the same results with theorem 5.1.

The key of solving this equation is the following lemma which can be easily proven.

**Lemma 7.1.** If  $y(x)$  is the solution of equation  $(D - \lambda_2 T_\alpha)y(x) = 0$ , it is also the solution of the equation  $(D - \lambda_1 T_\alpha)(D - \lambda_2 T_\alpha)y(x) = 0$ . If  $z(x)$  is the solution of equation  $(D - \lambda_1 T_\alpha)z(x) = 0$ , and  $y(x)$  is the solution of equation  $(D - \lambda_2 T_\alpha)y(x) = z(x)$ , then  $y(x)$  is also the solution of the equation  $(D - \lambda_1 T_\alpha)(D - \lambda_2 T_\alpha)y(x) = 0$ .

By the lemma 7.1, we have the following theorem.

**Theorem 7.2.** For the initial value problem at the original point  $x = 0$  of equation (68), there are the following two cases.

**Case (i).** If  $\lambda_1 \neq \lambda_2\alpha$ , the solution is give by

$$y(x) = \frac{c_1}{\frac{\lambda_1}{\alpha} - \lambda_2} E_\alpha\left(\frac{\lambda_1}{\alpha}x\right) + c_2 E_\alpha(\lambda_2 x),$$

where  $c_1$  and  $c_2$  can be determined by initial conditions.

**Case (ii).** If  $\lambda_1 = \lambda_2\alpha$ , the solution is give by

$$y(x) = c_1 E_\alpha(\lambda_1 x) + c_2 x E_\alpha(\alpha\lambda_1 x),$$

where  $c_1$  and  $c_2$  can be determined by initial conditions.

**Proof.** In fact, by lemma 7.1, the first basic solution  $y_1(x)$  satisfies

$$(D - \lambda_2 T_\alpha)y_1(x) = 0,$$

and then

$$y_1(x) = c_1 E_\alpha(\lambda_2 x).$$

Therefore, the second basic solution satisfies

$$(D - \lambda_2 T_\alpha)y(x) = c_2 E_\alpha(\lambda_1 x),$$

whose special solution is given by

$$y^*(x) = c_2 (D - \lambda_2 T_\alpha)^{-1} E_\alpha(\lambda_1 x).$$

From formula (67), if  $\lambda_1 \neq \lambda_2\alpha$ , we know

$$y^*(x) = \frac{c_2}{\frac{\lambda_1}{\alpha} - \lambda_2} E_\alpha\left(\frac{\lambda_1}{\alpha}x\right).$$

Hence, in this case, the general solution is given by

$$y(x) = c_1 E_\alpha(\lambda_2 x) + \frac{c_2}{\frac{\lambda_1}{\alpha} - \lambda_2} E_\alpha\left(\frac{\lambda_1}{\alpha}x\right),$$

where  $c_1$  and  $c_2$  are two arbitrary constants which can be determined by  $y(0)$  and  $y'(0)$ .

If  $\lambda_1 = \lambda_2\alpha$ , from formula (67),  $y(x)$  is meaningless, so this method is invalid. In this case, we need to solve

$$(D - \lambda_1 T_\alpha)y(x) = E_\alpha(\lambda_1 \alpha x). \quad (69)$$

By the power series method in section 4, we can assume that the solution has the following form

$$y(x) = c_1 E_\alpha(s_1 x) + c_2 x E_\alpha(s_2 x),$$

and insert it into the above equation to obtain

$$s_1 = \lambda_2, s_2 = \alpha \lambda_2 = \lambda_1.$$

So we have

$$y(x) = c_1 E_\alpha(\lambda_2 x) + c_2 x E_\alpha(\alpha \lambda_2 x),$$

where  $c_1$  and  $c_2$  can be determined by  $y(0)$  and  $y'(0)$ . The proof is completed.

In general, for the  $n$ -th order linear pantograph equation

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(\alpha t) + p_{n-2}y^{(n-2)}(\alpha^2 t) + \cdots + p_1 y'(\alpha^{n-1} t) + p_0 y(\alpha^n t) = 0, \quad (70)$$

we can write it as follows

$$D^n y(x) + p_{n-1} T_\alpha D^{n-1} y(x) + p_{n-2} T_\alpha^2 D^{n-2} y(x) \cdots + p_1 T_\alpha^{n-1} D y(x) + p_0 T_\alpha^n y(x) = 0,$$

where  $p'_i s$  are constants for  $i = 0, \dots, n-1$ , and  $T_\alpha^k = T_\alpha \cdots T_\alpha = T_{\alpha^k}$ . Furthermore, we have the following two ways to deal with it.

The first way is to solve it directly. By taking  $y(x) = E_\alpha(\beta t)$  and substituting it into the above equation and eliminating  $E_\alpha(\beta \alpha^n t)$ , we get the characteristic equation

$$\alpha^{\frac{n(n-1)}{2}} \beta^n + p_{n-1} \alpha^{\frac{(n-1)(n-2)}{2}} \beta^{n-1} + \cdots + p_1 \beta + p_0 = 0. \quad (71)$$

If this equation has no multiple roots, that is, it has  $n$  distinct roots  $\beta_j$  for  $j = 1, \dots, n$ , we can give the general solution by

$$y(t) = a_1 E_\alpha(\beta_1 t) + \cdots + a_n E_\alpha(\beta_n t), \quad (72)$$

where  $a_1, \dots, a_n$  can be uniquely determined by the initial conditions at the original point. If there are some multiple roots, we will use the corresponding operator method to deal with it. By the similar method, we can prove the following general theorem.

**Theorem 7.3.** If  $\beta_1, \dots, \beta_m$  are the distinct roots of the characteristic equation with multiplicities respectively  $n_1, \dots, n_m$  satisfying  $n_1 + \cdots + n_m = n$ . Then the following  $n$  functions make up the basic solutions system of the initial value problem at original point of the equation (70),

$$\begin{aligned} y_1(t) &= E_\alpha(\beta_1 t), y_2(t) = t E_\alpha(\beta_1 \alpha t), \dots, y_{n_1}(t) = t^{n_1-1} E_\alpha(\beta_1 \alpha^{n_1-1} t), \\ y_{n_1+1}(t) &= E_\alpha(\beta_2 t), \dots, y_{n_1+n_2}(t) = t^{n_2-1} E_\alpha(\beta_2 \alpha^{n_2-1} t), \end{aligned}$$

...

$$y_{n-n_m+1}(t) = E_\alpha(\beta_m t), \dots, y_n(t) = t^{n_m-1} E_\alpha(\beta_m \alpha^{n_m-1} t). \quad (73)$$

In other words, the solution of the initial value problem at original point of the equation (70) is the linear combination of these basic solutions with coefficients which can be determined by initial conditions.

**Proof.** We only take  $n = 3$  and suppose that  $\lambda_0$  is a characteristic root with multiplicity three to prove the result. The general case can be proved by the similar method. Denote  $Q_3(D, T_\alpha)$  and  $Q_3(\lambda)$  respectively as

$$Q_3(D, T_\alpha) = D^3 + p_2 T_\alpha D^2 + p_1 T_\alpha^2 D + p_0 T_\alpha^3,$$

$$Q_3(\lambda) = \alpha^3 \lambda^3 + p_2 \alpha \lambda^2 + p_1 \lambda + p_0.$$

By direct computation, we have

$$\begin{aligned} Q_3(D, T_\alpha)(x^2 E_\alpha(\lambda_0 \alpha^2 x)) &= Q_3(\lambda_0) \alpha^6 x^2 E_\alpha(\lambda_0 \alpha^5 x) \\ &+ Q'_3(\lambda_0) 2\alpha x E_\alpha(\lambda_0 \alpha^4 x) + Q''_3(\lambda_0) E_\alpha(\lambda_0 \alpha^3 x). \end{aligned}$$

Furthermore, since  $\lambda_0$  is a characteristic root with multiplicity three, we have

$$Q_3(\lambda_0) = Q'_3(\lambda_0) = Q''_3(\lambda_0) = 0,$$

and hence, it follows that

$$Q_3(D, T_\alpha)(x^2 E_\alpha(\lambda_0 \alpha^2 x)) = 0.$$

This means that  $x^2 E_\alpha(\lambda_0 \alpha^2 x)$  is a basic solution. By the same reason,  $x E_\alpha(\lambda_0 \alpha x)$  and  $E_\alpha(\lambda_0 x)$  are other two basic solutions. Therefore, the general solution can be given by

$$y(x) = c_1 E_\alpha(\lambda_0 x) + c_2 x E_\alpha(\lambda_0 \alpha x) + c_3 x^2 E_\alpha(\lambda_0 \alpha^2 x), \quad (74)$$

where  $c_1, c_2$  and  $c_3$  can be determined by the initial conditions. The proof is completed.

The second way is to transform Eq.(70) into the system of linear pantograph equations. In fact, letting  $x_1(t) = y(t), x_2(t) = x'_1(\frac{t}{\alpha}), \dots, x_n(t) = x'_{n-1}(\frac{t}{\alpha})$ , then the equation (70) is transformed into the following system of the first order linear equations

$$x'_1(t) = x_2(\alpha t),$$

$$x'_2(t) = x_3(\alpha t),$$

...

$$x'_n(t) = -\frac{p_0}{\alpha^{n-1}} x_1(\alpha t) - \frac{p_1}{\alpha^{n-1}} x_2(\alpha t) - \dots - \frac{p_{n-1}}{\alpha^{n-1}} x_n(\alpha t). \quad (75)$$

For example, the second order pantograph equation

$$q''(t) = -kq(\alpha^2 t),$$



can be transformed to the Hamiltonian-like form

$$\begin{aligned} q'(t) &= p(\alpha t), \\ p'(t) &= -\frac{k}{\alpha}q(\alpha t). \end{aligned}$$

This means that for the initial value problem at original point  $t = 0$ , the theory of the high order linear pantograph equations is equivalent to the theory of the system of the first order linear pantograph equations. Therefore, we can give the solutions of the high order pantograph equation by the solution of the system of the first order pantograph equations.

For simplicity, we denote

$$P(D, T_\alpha) = D^n + p_{n-1}T_\alpha D^{n-1} + p_{n-2}T_\alpha^2 D^{n-2} \cdots + p_1 T_\alpha^{n-1} D + p_0 T_\alpha^n.$$

We have the following result on the non-homogenous equation.

**Theorem 7.4.** For non-homogenous equation

$$P(D, T_\alpha)y(x) = \sum_{k=1}^m A_k E_\alpha(r_k x), \quad (76)$$

where  $A_k$  and  $r_k$  are known constants, a special solution is given by

$$y^*(x) = \sum_{k=1}^m B_k E_\alpha\left(\frac{r_k}{\alpha^n} x\right), \quad (77)$$

where

$$B_k = \frac{A_k}{\sum_{j=0}^n p_j r_k^j \alpha^{-\frac{j(2n-j+1)}{2}}}, \quad (78)$$

where  $p_n = 1$  and we require

$$\sum_{j=0}^n p_j r_k^j \alpha^{-\frac{j(2n-j+1)}{2}} \neq 0. \quad (79)$$

Further, the general solution is given by the summation of the general solution of the homogenous equation and the special solution.

**Proof.** Assume that the special solution of the equation  $P(D, T_\alpha)y(x) = A_k E_\alpha(r_k x)$  is  $y_k(x) = B_k E_\alpha(s_k x)$ . Then, we have

$$y_k^{(j)}(x) = B_k s_k^j \alpha^{\frac{j(j-1)}{2}} E_\alpha(\alpha^j s_k x),$$

and substitute it into the equation to get  $s_k = \frac{r_k}{\alpha^n}$  and the values of  $B_k$ . The proof is completed.

When we take  $n = 2$ , we get the theorem 5.2.

**Definition 7.1.** If  $r_k$  satisfies the equation (79), it is called the resonance frequency, correspondingly,  $A_k E_\alpha(r_k x)$  is called the resonance term.

Now we consider the non-homogenous linear pantograph equations with resonance terms. We only give the result on the second order equation. Other high order equation can be dealt with by the same method.

**Theorem 7.5.** Denote  $Q(r) = r^2 + pr\alpha + q\alpha^3$ . For the non-homogenous pantograph equation

$$y''(x) + py'(\alpha x) + qy(\alpha^2 x) = AE_\alpha(rx),$$

where  $Q(r) \neq 0$  which means that  $AE_\alpha(rx)$  is a resonance term. There are the following two cases:

**Case (i).**  $Q(r) = 0$  and  $Q'(r) = 2r + p\alpha \neq 0$ . Then a special solution is given by

$$y(x) = \frac{A\alpha}{2r + p\alpha} x E_\alpha\left(\frac{r}{\alpha}x\right). \quad (80)$$

**Case (ii).**  $Q(r) = 0$  and  $Q'(r) = 2r + p\alpha = 0$ . Then a special solution is given by

$$y(x) = \frac{A}{2} x^2 E_\alpha(rx). \quad (81)$$

**Proof.** Firstly, we prove the result in case (i). Assume that the special solution has the form

$$y(x) = Bx E_\alpha(sx).$$

Then we have

$$\begin{aligned} y'(x) &= B E_\alpha(sx) + Bsx E_\alpha(s\alpha x), \\ y''(x) &= 2Bs E_\alpha(s\alpha x) + Bs^2 \alpha x E_\alpha(s\alpha^2 x). \end{aligned}$$

Substituting these terms into the equation give  $Q(r) = 0$  and

$$s = \frac{r}{\alpha}, B = \frac{A\alpha}{2r + p\alpha}.$$

Next we prove the case (ii). Assume that the special solution has the form

$$y(x) = Bx^2 E_\alpha(sx).$$

Then we have

$$\begin{aligned} y'(x) &= 2Bx E_\alpha(sx) + Bsx^2 E_\alpha(s\alpha x), \\ y''(x) &= 2B E_\alpha(sx) + 4Bsx E_\alpha(s\alpha x) + Bs^2 \alpha x^2 E_\alpha(s\alpha^2 x). \end{aligned}$$

Substituting these terms into the equation give  $Q(r) = 0$  and  $Q'(r) = 0$  and

$$s = r, B = \frac{A}{2}.$$

The proof is completed.

**Remark 7.1.** The theorems 7.4 and 7.5 only mean that if the solution exists, it will be given by the summation of the special solution and the general solution of the homogenous equation, but it doesn't mean that the solution of the non-homogenous equation (76) must exist. The existence, uniqueness and non-uniqueness of the non-homogenous pantograph equations will be discussed in section 8.

## 8 Existence, uniqueness, non-uniqueness and representation of solutions at a general initial point

In previous sections, all results are obtained under the initial conditions at the original point. If the initial condition is taken at a general point which is not the original point, whether do the existence and uniqueness of solution hold? How to represent these solutions if they exist? How many solutions will exist if uniqueness does not hold? These problems are not trivial. In the section, we will give the results about these problems. For the purpose, we need an important theorem about the zeroes of  $E_\alpha(x)$  which is perhaps considered as the most important property of the function which is equivalent to the theorem 6 in [9].

**Morris-Feldstein-Bowen-Hahn Theorem: first form[9].** Every non-trivial solution of the equation  $y'(x) = -y(\alpha x)$  has an infinity of positive zeroes.

Equivalently, in other words, we write it as the following form by using the special function  $E_\alpha(x)$ .

**Morris-Feldstein-Bowen-Hahn Theorem: second form.**(MFBH theorem for simplicity).  $E_\alpha(x)$  has an infinity of negative zeros.

**Remark 8.1.** The above theorem is obtained and proven in [9] by Morris-Feldstein-Bowen, and an elementary proof is given by Hahn [9].

By the MFBH theorem, we can get the following theorem.

**Theorem 8.1.** For the initial value problem of linear equation

$$y'(x) = ky(\alpha x), y(x_0) = y_0, \quad (82)$$

where  $x_0 \neq 0$ , we have the following results.

(i). If  $x_0$  is not the zero of  $E_\alpha(kx)$ , that is,  $E_\alpha(kx_0) \neq 0$ , then there exists the unique solution

$$y(x) = \frac{y_0 E_\alpha(kx)}{E_\alpha(kx_0)}. \quad (83)$$

(ii). If  $y_0 = 0$  and  $x_0$  is the zero of  $E_\alpha(kx)$ , that is,  $E_\alpha(kx_0) = 0$ , then there exist an infinity of solutions all of which can be represented by

$$y(x) = cE_\alpha(kx), \quad (84)$$

where  $c$  is an arbitrary constant.

(iii). If  $y_0 \neq 0$  and  $E_\alpha(kx_0) = 0$ , then there does not exist solution.

**Proof.** For case (i), it is easy to see that  $y(x) = \frac{y_0 E_\alpha(kx)}{E_\alpha(kx_0)}$  is a solution for the initial value problem. Then we only need to prove that it is the unique solution. Assume that  $y_1$  and  $y_2$  are two solutions satisfying  $y_1(x_0) = y_2(x_0) = y_0$ . Letting  $z(x) = y_1(x) - y_2(x)$ , then we have

$$z'(x) = kz(\alpha x), z(x_0) = 0.$$

If we can prove  $z(0) = 0$ , then we can get  $z(x) \equiv 0$  by the unique theorem 2.1 at the origin point. In fact, if  $z(0) = z_0 \neq 0$ , by the unique theorem 2.1 and

proposition 3.1, we must get

$$z(x) = z_0 E_\alpha(kx),$$

and hence

$$z(x_0) = z_0 E_\alpha(kx_0) \neq 0,$$

which is contradictory to  $z(x_0) = 0$ .

For the case (ii), the existence of the solution is obvious. Let  $y(x)$  be any solution with  $y(x_0) = 0$ , then  $y(x)$  has a value  $y(0)$  at  $x = 0$ , and hence we have a solution  $y(x) = y(0)E_\alpha(kx)$ . By taking  $y(0) = c$ , we get the conclusion. The case (iii) is obvious. The proof is completed.

**Theorem 8.2.** For existence and uniqueness of the solution to the initial value problem of non-homogenous equation

$$y'(x) = \lambda y(\alpha x) + qE_\alpha(\lambda \alpha x), y(x_0) = y_0, \quad (85)$$

there are the following three cases:

(i). If  $E_\alpha(\lambda x_0) \neq 0$ , then there exists the unique solution

$$y(x) = \frac{y_0 - qx_0 E_\alpha(\lambda \alpha x_0)}{E_\alpha(\lambda x_0)} E_\alpha(\lambda x) + qx E_\alpha(\lambda \alpha x). \quad (86)$$

(ii). If  $E_\alpha(\lambda x_0) = 0$  and  $y_0 = qx_0 E_\alpha(\lambda \alpha x_0)$ , then there exist an infinity of solutions all of which can be given by

$$y(x) = cE_\alpha(\lambda x) + qx E_\alpha(\lambda \alpha x), \quad (87)$$

where  $c$  is an arbitrary constant.

(iii). If  $E_\alpha(\lambda x_0) = 0$  and  $y_0 \neq qx_0 E_\alpha(\lambda \alpha x_0)$ , then there does not exist solution.

**Proof.** (i). We only need to prove the uniqueness. Assuming that  $z(x)$  is another solution with  $z(0) = c$ , by theorem 4.2, we have

$$z(x) = cE_\alpha(\lambda x) + qx E_\alpha(\lambda \alpha x).$$

Since we also require  $z(x_0) = y_0$ , so we get  $c = \frac{y_0 - qx_0 E_\alpha(\lambda \alpha x_0)}{E_\alpha(\lambda x_0)}$ , and then  $y(x) = z(x)$ .

(ii). Firstly, we can verify that (87) is just solution and satisfies these two conditions. Hence we only need to prove that any solution satisfying these conditions has the form of (87). Assume that  $y(x)$  is the solution satisfying these two conditions. Then, according to the value  $y(0)$  of  $y(x)$  at  $x = 0$ , we can use the theorem 4.2 to give the solution

$$y(x) = y(0)E_\alpha(\lambda x) + qx E_\alpha(\lambda \alpha x).$$

Letting  $y(0) = c$  gives conclusion.

(iii). It is obvious. The proof is completed.

**Theorem 8.3.** For the linear equations system

$$y_1'(x) = \lambda y_1(\alpha x) + y_2(\alpha x), \quad (88)$$

$$y_2'(x) = \lambda y_2(\alpha x), \quad (89)$$

with initial values  $y_1(x_0)$  and  $y_2(x_0)$ , there are the following five cases about its solution.

(i). If  $E_\alpha(\lambda x_0) \neq 0$ , then there exists the unique solution

$$y_1(x) = \frac{y_1(x_0) - \frac{y_2(x_0)}{E_\alpha(\lambda x_0)} x_0 E_\alpha(\lambda \alpha x_0)}{E_\alpha(\lambda x_0)} E_\alpha(\lambda x) + \frac{y_2(x_0) x E_\alpha(\lambda \alpha x)}{E_\alpha(\lambda x_0)}, \quad (90)$$

$$y_2(x) = \frac{y_2(x_0)}{E_\alpha(\lambda x_0)} E_\alpha(\lambda x). \quad (91)$$

(ii). If  $E_\alpha(\lambda x_0) = 0$ ,  $y_2(x_0) = 0$  and  $E_\alpha(\lambda \alpha x_0) \neq 0$ , then there exist an infinity of solutions all of which can be given by

$$y_1(x) = c E_\alpha(\lambda x) + \frac{y_1(x_0) x E_\alpha(\lambda \alpha x)}{x_0 E_\alpha(\lambda \alpha x_0)}, \quad (92)$$

$$y_2(x) = \frac{y_1(x_0)}{x_0 E_\alpha(\lambda \alpha x_0)} E_\alpha(\lambda x), \quad (93)$$

where  $c$  is an arbitrary constant.

(iii). If  $E_\alpha(\lambda x_0) = 0$ ,  $y_2(x_0) = 0$  and  $E_\alpha(\lambda \alpha x_0) = 0$  and  $y_1(x_0) = 0$ , then there exist an infinity of solutions all of which can be given by

$$y_1(x) = c_1 E_\alpha(\lambda x) + c_2 x E_\alpha(\lambda \alpha x), \quad (94)$$

$$y_2(x) = c_2 E_\alpha(\lambda x), \quad (95)$$

where  $c_1$  and  $c_2$  are two arbitrary constants.

(iv). If  $E_\alpha(\lambda x_0) = 0$ ,  $y_2(x_0) = 0$  and  $E_\alpha(\lambda \alpha x_0) = 0$  and  $y_1(x_0) \neq 0$ , then there does not exist solution.

(v). If  $E_\alpha(\lambda x_0) = 0$  and  $y_2(x_0) \neq 0$ , then there does not exist solutions.

**Proof.** By theorem 8.1 and theorem 8.2, we can easily prove it. The proof is completed.

Now we consider the second order pantograph equation

$$y''(t) + p y'(\alpha t) + q y(\alpha^2 t) = 0, \quad (96)$$

with initial conditions

$$y(t_0) = A, y'(\frac{t_0}{\alpha}) = B, \quad (97)$$

where  $t_0$  is a general point. Letting  $x_1(t) = y(t)$  and  $x_2(t) = y'(\frac{t}{\alpha})$ , we transform the second order equation into the system of the first order equations as follows

$$x_1'(t) = x_2(\alpha t), \quad (98)$$

$$x_2'(t) = -\frac{q}{\alpha}x_1(\alpha t) - \frac{p}{\alpha}x_2(\alpha t), \quad (99)$$

with initial conditions

$$x_1(t_0) = A, x_2(t_0) = B. \quad (100)$$

For the coefficients matrix

$$K = \begin{pmatrix} 0 & 1 \\ -\frac{q}{\alpha} & -\frac{p}{\alpha} \end{pmatrix}, \quad (101)$$

when  $p^2 \neq 4q\alpha$ , there exist two distinct eigenvalues  $\lambda_1 \neq \lambda_2$ , then there is an invertible matrix  $P$  such that

$$P^{-1}KP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \quad (102)$$

When  $p^2 = 4q\alpha$ , we have  $\lambda_1 = \lambda_2 = \lambda = -\frac{p}{2\alpha}$ , and the corresponding elementary factor is  $(\lambda + \frac{p}{2\alpha})^2$ , and hence there is an invertible matrix  $P$  such that

$$P^{-1}KP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \quad (103)$$

Denote

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad (104)$$

$$P^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad (105)$$

and

$$x_1(t) = p_{11}z_1(t) + p_{12}z_2(t), \quad (106)$$

$$x_2(t) = p_{21}z_1(t) + p_{22}z_2(t). \quad (107)$$

According to the theorems 8.1-8.3, we can prove the following result.

**Theorem 8.4.** For the system of equations (98) and (99) with condition (100), there are the following two cases to be discussed:

**Case 1.** When  $p^2 \neq 4q\alpha$ , then the matrix  $K$  has two distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , and hence we have

$$z_1'(t) = \lambda_1 z_1(\alpha t), \quad (108)$$

$$z_2'(t) = \lambda_2 z_2(\alpha t), \quad (109)$$

with initial conditions

$$z_1(t_0) = q_{11}x_1(t_0) + q_{12}x_2(t_0) = q_{11}A + q_{12}B, \quad (110)$$

$$z_2(t_0) = q_{21}x_1(t_0) + q_{22}x_2(t_0) = q_{21}A + q_{22}B. \quad (111)$$

Then there are the following six cases:

(i). If  $E_\alpha(\lambda_1 t_0) \neq 0$  and  $E_\alpha(\lambda_2 t_0) \neq 0$ , then there exists the unique solution

$$y(t) = \frac{p_{11}(q_{11}A + q_{12}B)}{E_\alpha(\lambda_1 t_0)} E_\alpha(\lambda_1 t) + \frac{p_{12}(q_{21}A + q_{22}B)}{E_\alpha(\lambda_2 t_0)} E_\alpha(\lambda_2 t), \quad (112)$$

or equivalently

$$y(t) = \frac{A\lambda_2 - B}{\lambda_2 - \lambda_1} \frac{E_\alpha(\lambda_1 t)}{E_\alpha(\lambda_1 t_0)} + \frac{A\lambda_1 - B}{\lambda_1 - \lambda_2} \frac{E_\alpha(\lambda_2 t)}{E_\alpha(\lambda_2 t_0)}. \quad (113)$$

(ii). If  $E_\alpha(\lambda_1 t_0) \neq 0$ ,  $E_\alpha(\lambda_2 t_0) = 0$  and  $z_2(t_0) = 0$ , then there exist an infinity of solutions all of which can be given by

$$y(t) = p_{11} \frac{q_{11}A + q_{12}B}{E_\alpha(\lambda_1 t_0)} E_\alpha(\lambda_1 t) + p_{12}c E_\alpha(\lambda_2 t), \quad (114)$$

or equivalently

$$y(t) = \frac{AE_\alpha(\lambda_1 t)}{E_\alpha(\lambda_1 t_0)} + cE_\alpha(\lambda_2 t), \quad (115)$$

where  $c$  is an arbitrary constant.

(iii). If  $E_\alpha(\lambda_2 t_0) \neq 0$ ,  $E_\alpha(\lambda_1 t_0) = 0$  and  $z_1(t_0) = 0$ , then there exist an infinity of solutions all of which can be given by

$$y(t) = p_{12} \frac{q_{21}A + q_{22}B}{E_\alpha(\lambda_2 t_0)} E_\alpha(\lambda_2 t) + p_{11}c E_\alpha(\lambda_1 t), \quad (116)$$

or equivalently

$$y(t) = \frac{AE_\alpha(\lambda_2 t)}{E_\alpha(\lambda_2 t_0)} + cE_\alpha(\lambda_1 t), \quad (117)$$

where  $c$  is an arbitrary constant.

(iv). If  $E_\alpha(\lambda_1 t_0) = 0$  and  $z_1(t_0) = 0$ ,  $E_\alpha(\lambda_2 t_0) = 0$  and  $z_2(t_0) = 0$ , then  $A = B = 0$ , and there exist an infinity of solutions all of which can be given by

$$y(t) = c_1 E_\alpha(\lambda_1 t) + c_2 E_\alpha(\lambda_2 t), \quad (118)$$

where  $c_1$  and  $c_2$  are two arbitrary constants.

(v). If  $E_\alpha(\lambda_1 t_0) = 0$  and  $z_1(t_0) \neq 0$ , then there does not exist solution.

(vi). If  $E_\alpha(\lambda_2 t_0) = 0$  and  $z_2(t_0) \neq 0$ , then there does not exist solution.

**Case 2.** When  $p^2 = 4q\alpha$ , we have  $\lambda_1 = \lambda_2 = \lambda = -\frac{p}{2\alpha}$ , then there are the following five cases:

(i). If  $E_\alpha(\lambda t_0) \neq 0$ , then there exists the unique solution

$$y(t) = p_{11} \left\{ \frac{z_1(t_0)E_\alpha(\lambda t_0) - z_2(t_0)t_0 E_\alpha(\lambda \alpha t_0)}{E_\alpha^2(\lambda t_0)} E_\alpha(\lambda t) + \frac{z_2(t_0)t E_\alpha(\lambda \alpha t)}{E_\alpha(\lambda t_0)} \right\} + p_{12} \frac{z_2(t_0)E_\alpha(\lambda t)}{E_\alpha(\lambda t_0)},$$

or equivalently,

$$y(t) = \frac{AE_\alpha(\lambda t_0) - (B - A\lambda)t_0 E_\alpha(\lambda \alpha t_0)}{E_\alpha^2(\lambda t_0)} E_\alpha(\lambda t) + \frac{B - A\lambda}{E_\alpha(\lambda t_0)} t E_\alpha(\lambda \alpha t). \quad (119)$$

(ii). If  $E_\alpha(\lambda t_0) = 0$ ,  $E_\alpha(\lambda \alpha t_0) \neq 0$  and  $z_2(t_0) = 0$ , then there exist an infinity of solutions all of which can be given by

$$y(t) = p_{11}\{cE_\alpha(\lambda t) + \frac{z_1(t_0)tE_\alpha(\lambda \alpha t)}{t_0E_\alpha(\lambda \alpha t_0)}\} + p_{12}\frac{z_1(t_0)E_\alpha(\lambda t)}{t_0E_\alpha(\lambda \alpha t_0)},$$

or equivalently,

$$y(t) = cE_\alpha(\lambda t) + \frac{A}{t_0E_\alpha(\lambda \alpha t_0)}tE_\alpha(\lambda \alpha t), \quad (120)$$

where  $c$  is an arbitrary constant.

(iii). If  $E_\alpha(\lambda t_0) = 0$ ,  $E_\alpha(\lambda \alpha t_0) = 0$  and  $z_1(t_0) = z_2(t_0) = 0$ , then  $A = B = 0$ , and there exist an infinity of solutions all of which can be given by

$$y(t) = c_1E_\alpha(\lambda t) + c_2tE_\alpha(\lambda \alpha t), \quad (121)$$

where  $c_1$  and  $c_2$  are two arbitrary constants.

(iv). If  $E_\alpha(\lambda t_0) = 0$ ,  $E_\alpha(\lambda \alpha t_0) = 0$  and  $z_2(t_0) = 0$  and  $z_1(t_0) \neq 0$ , then there does not exist solutions.

(v). If  $E_\alpha(\lambda t_0) = 0$  and  $z_2(x_0) \neq 0$ , then there does not exist solutions.  
**Proof.** We only consider (ii) in the case 1 and give a detailed proof of the equivalence of (110) and (111). Other cases can be proved similarly. Then, we only need to prove  $p_{11}(q_{11}A + q_{12}B) = A$ . In fact, since  $\lambda_1$  is the first eigenvalue of the matrix  $K$ , we have  $p_{21} = \lambda_1 p_{11}$ . Further, by  $P^{-1}P = E$ , it follows that  $p_{11}q_{12} + p_{12}q_{22} = 0$  and  $p_{11}q_{11} + p_{21}q_{12} = 1$ . Therefore, from  $z_2(t_0) = 0$ , that is,  $q_{21}A + q_{22}B = 0$ . we get

$$\frac{B}{A} = -\frac{q_{21}}{q_{22}} = \frac{p_{21}}{p_{11}} = \lambda_1.$$

And hence, we have

$$p_{11}(q_{11}A + q_{12}B) = A(p_{11}q_{11} + \lambda_1 p_{11}q_{12}) = A(p_{11}q_{11} + p_{21}q_{12}) = A.$$

The proof is completed.

These above theorems are the basic results for the existence and uniqueness of the kind of pantograph equations. It is easy to generalize the theorems 8.3 and 8.4 to  $n$  dimensional case, but the result is complicated, so we do not write it. By combining these results with the theorem 6.2 and theorem 7.3, we can easily give the corresponding existence and uniqueness results for the initial value problems at a general point of linear pantograph equations system and high order linear pantograph equations. For simplicity, we also do not write them. However, we must point out that at a general initial point, the solutions of the initial value problem are complicated, and includes no solution, unique solution and an infinity of solutions. All of these are rooted in the MFBH theorem. These results show the essential differences between the pantograph equation and usual ordinary differential equations.

**Remark 8.2.** We must have noticed that for the system of the second order linear pantograph equations (98) and (99), its initial conditions are very special,



that is, conditions (100) of  $y$  and  $y'$  are taken respectively at two points  $t_0$  and  $\frac{t_0}{\alpha}$  but not the only one point  $t_0$ . This is because we can use the equivalent linear pantograph equations system to deal with it. If the initial point is taken at only point  $t_0$ , we can use another method to give the following result.

**Theorem 8.5.** Consider the second order linear pantograph equation

$$y''(t) + py'(\alpha t) + qy(\alpha^2 t) = 0, \quad (122)$$

with initial conditions

$$y(t_0) = A, y'(t_0) = B, \quad (123)$$

where  $t_0$  is a general point. Let  $\lambda_1$  and  $\lambda_2$  be two roots of its characteristic equation

$$\alpha\lambda^2 + p\lambda + q = 0. \quad (124)$$

Then we have the following two cases:

**Case 1.**  $p^2 \neq 4\alpha q$ , that is  $\lambda_1 \neq \lambda_2$ . There are three cases for the solutions.

(i). If  $\begin{vmatrix} E_\alpha(\lambda_1 t_0) & E_\alpha(\lambda_2 t_0) \\ \lambda_1 E_\alpha(\alpha \lambda_1 t_0) & \lambda_2 E_\alpha(\alpha \lambda_2 t_0) \end{vmatrix} \neq 0$ , then there is the unique solution

$$y(t) = c_1 E_\alpha(\lambda_1 t) + c_2 E_\alpha(\lambda_2 t), \quad (125)$$

where  $c_1$  and  $c_2$  can be determined by the following equations system

$$c_1 E_\alpha(\lambda_1 t_0) + c_2 E_\alpha(\lambda_2 t_0) = A, \quad (126)$$

$$c_1 \lambda_1 E_\alpha(\alpha \lambda_1 t_0) + c_2 \lambda_2 E_\alpha(\alpha \lambda_2 t_0) = B. \quad (127)$$

(ii). If  $\begin{vmatrix} E_\alpha(\lambda_1 t_0) & E_\alpha(\lambda_2 t_0) \\ \lambda_1 E_\alpha(\alpha \lambda_1 t_0) & \lambda_2 E_\alpha(\alpha \lambda_2 t_0) \end{vmatrix} = 0$ , and the rank of the matrix  $\begin{pmatrix} E_\alpha(\lambda_1 t_0) & E_\alpha(\lambda_2 t_0) & A \\ \lambda_1 E_\alpha(\alpha \lambda_1 t_0) & \lambda_2 E_\alpha(\alpha \lambda_2 t_0) & B \end{pmatrix}$  is 1, then there is an infinity of solutions all of which can be given by

$$y(t) = c_1 E_\alpha(\lambda_1 t) + c_2 E_\alpha(\lambda_2 t), \quad (128)$$

where  $c_1$  and  $c_2$  satisfy (126) and (127) which has an infinity of solutions.

(iii). If  $\begin{vmatrix} E_\alpha(\lambda_1 t_0) & E_\alpha(\lambda_2 t_0) \\ \lambda_1 E_\alpha(\alpha \lambda_1 t_0) & \lambda_2 E_\alpha(\alpha \lambda_2 t_0) \end{vmatrix} = 0$ , and the rank of the matrix  $\begin{pmatrix} E_\alpha(\lambda_1 t_0) & E_\alpha(\lambda_2 t_0) & A \\ \lambda_1 E_\alpha(\alpha \lambda_1 t_0) & \lambda_2 E_\alpha(\alpha \lambda_2 t_0) & B \end{pmatrix}$  is 2, then there does not exist solution.

**Case 2.**  $p^2 = 4\alpha q$ , that is  $\lambda_1 = \lambda_2 = \lambda$ . There are three cases for the solutions.

(i). If  $\begin{vmatrix} E_\alpha(\lambda t_0) & t_0 E_\alpha(\lambda \alpha t_0) \\ \lambda E_\alpha(\alpha \lambda t_0) & E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0) \end{vmatrix} \neq 0$ , then there is the unique solution

$$y(t) = c_1 E_\alpha(\lambda t) + c_2 t E_\alpha(\lambda \alpha t), \quad (129)$$

where  $c_1$  and  $c_2$  can be determined by the following equations system

$$c_1 E_\alpha(\lambda t_0) + c_2 t_0 E_\alpha(\lambda \alpha t_0) = A, \quad (130)$$

$$c_1 \lambda E_\alpha(\alpha \lambda t_0) + c_2 (E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0)) = B. \quad (131)$$

(ii). If  $\begin{vmatrix} E_\alpha(\lambda t_0) & t_0 E_\alpha(\lambda \alpha t_0) \\ \lambda E_\alpha(\alpha \lambda t_0) & E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0) \end{vmatrix} = 0$ , and the rank of the matrix  $\begin{pmatrix} E_\alpha(\lambda t_0) & t_0 E_\alpha(\lambda \alpha t_0) & A \\ \lambda E_\alpha(\alpha \lambda t_0) & E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0) & B \end{pmatrix}$  is 1, then there is an infinity of solutions all of which can be given by

$$y(t) = c_1 E_\alpha(\lambda_1 t) + c_2 E_\alpha(\lambda_2 t), \quad (132)$$

where  $c_1$  and  $c_2$  satisfy (130) and (131) which has an infinity of solutions.

(iii). If  $\begin{vmatrix} E_\alpha(\lambda t_0) & t_0 E_\alpha(\lambda \alpha t_0) \\ \lambda E_\alpha(\alpha \lambda t_0) & E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0) \end{vmatrix} = 0$ , and the rank of the matrix  $\begin{pmatrix} E_\alpha(\lambda t_0) & t_0 E_\alpha(\lambda \alpha t_0) & A \\ \lambda E_\alpha(\alpha \lambda t_0) & E_\alpha(\alpha \lambda t_0) + \lambda \alpha t_0 E_\alpha(\alpha^2 \lambda t_0) & B \end{pmatrix}$  is 2, then there does not exist solution.

**Proof.** We only prove the case 1, the case 2 can be proven similarly. In fact, since the solution and its derivative can take values at the point  $t = 0$ , we can get the general solution as

$$y(t) = c_1 E_\alpha(\lambda_1 t) + c_2 E_\alpha(\lambda_2 t).$$

Therefore, we have

$$A = y(t_0) = c_1 E_\alpha(\lambda_1 t_0) + c_2 E_\alpha(\lambda_2 t_0),$$

$$B = y'(t_0) = c_1 \lambda_1 E_\alpha(\lambda_1 \alpha t_0) + c_2 \lambda_2 E_\alpha(\lambda_2 \alpha t_0).$$

According to the theory of linear algebraic equations system, we get the corresponding conclusions (i-iii) of case 1. The proof is completed.

**Remark 8.3.** We have seen that for general linear homogenous pantograph equations, we can write their general solutions. When we consider the initial value problem at a general point, we will need to determine these constants in general solutions. However, since exponent-like function  $E_\alpha(x)$  has an infinity of real zeroes, we can't find these coefficients or can find an infinite number of solutions in some cases. If we further consider the existence, uniqueness and non-uniqueness of the solution of the initial value problem at a general point for the non-homogenous linear pantograph equations, we also obtain the similar results by the similar consideration.

In the theorem 8.2. we have considered the first order non-homogenous linear pantograph equation with the resonance term. Next, we give the result in the non-resonance case. Other cases such as the second order linear non-homogenous pantograph equation can be dealt with by the similar method, and the results can be derived from the corresponding results on the homogenous equations such as the theorems 8.5 and 7.5.

**Theorem 8.6.** For the non-homogenous pantograph equation

$$y'(x) + \beta y(\alpha x) = A E_\alpha(r x), y(x_0) = y_0,$$

where  $x_0 \neq 0$  and  $r \neq \alpha\beta$ . There are the following three cases:

(i). If  $E_\alpha(\beta x_0) \neq 0$ , then there exists the unique solution

$$y(x) = \frac{y_0 - \frac{A\alpha}{r-\alpha\beta}E_\alpha(\frac{r}{\alpha}x_0)}{E_\alpha(\beta x_0)}E_\alpha(\beta x) + \frac{A\alpha}{r-\alpha\beta}E_\alpha(\frac{r}{\alpha}x). \quad (133)$$

(ii). If  $E_\alpha(\beta x_0) = 0$  and  $y_0 = \frac{A\alpha}{r-\alpha\beta}E_\alpha(\frac{r}{\alpha}x_0)$ , then there exist an infinity of solutions all of which can be given by

$$y(x) = cE_\alpha(\beta x) + \frac{A\alpha}{r-\alpha\beta}E_\alpha(\frac{r}{\alpha}x), \quad (134)$$

where  $c$  is an arbitrary constant.

(iii). If  $E_\alpha(\beta x_0) = 0$  and  $y_0 \neq \frac{A\alpha}{r-\alpha\beta}E_\alpha(\frac{r}{\alpha}x_0)$ , then there does not exist the solution.

**Proof.** According to the general solution (67), we can easily prove it. The proof is completed.

Finally, we give a generalization of the MFBH theorem to the first order linear pantograph equation with variable coefficient. By the similar method of proving MFBH theorem by Hahn[9], we give the proofs of the following two results.

**Theorem 8.7.** For the equation (here  $0 < \alpha < 1$ )

$$y'(t) = -k(t)y(\alpha t), y(0) \neq 0, \quad (135)$$

if  $k'(t) \leq 0$  and  $k(t) > k_0$  where  $k_0 > 0$  is a constant, then the solution has an infinity of positive zeros.

**Proof.** We will prove that it is impossible that  $y(t) > 0$  for all  $t > 0$ . Suppose that  $y(t) > 0$  for all  $t > 0$ , then  $y(t) > 0$  for  $t > \alpha^2 t_0$  where  $t_0 > 0$  is an arbitrary constant. Therefore, for  $t > t_0$ , we have

$$\begin{aligned} y(t) &> 0, y'(t) < 0, \\ y''(t) &= -k'(t)y(\alpha t) + \alpha k(t)k(\alpha t)y(\alpha^2 t) > 0. \end{aligned}$$

Take a sequence of points  $t_n = \frac{t_0}{\alpha^n}$  for  $n = 0, 1, \dots$ . On the interval  $[t_n, t_{n+1}]$ , the graph of the function  $y(t)$  is convex and decreasing, so the arc  $A_n A_{n+1}$  lies above the tangent line  $L$  at point  $A_{n+1} = (t_{n+1}, y(t_{n+1}))$  and below the line through the point  $A_n = (t_n, y(t_n))$  parallel to  $L$ . And then, we have

$$y(t_{n+1}) < y(t_n)(1 - k(t_n)(t_{n+1} - t_n)) = y(t_n)(1 - \frac{k(t_n)}{\alpha^n}(t_1 - t_0)).$$

Since  $k(t) > k_0 > 0$  and  $0 < \alpha < 1$ , we have  $\frac{k(t_n)}{\alpha^n} \rightarrow +\infty$  as  $n \rightarrow +\infty$ , and hence  $1 - \frac{k(t_n)}{\alpha^n}(t_1 - t_0)$  will become negative for sufficiently large  $n$ . This is a contradiction with the assumption. By the similar reason,  $y(t)$  can not be negative for sufficient large  $t > 0$ . The proof is completed.

Similarly, we have the following result.

**Theorem 8.8.** For the equation (here  $0 < \alpha < 1$ )

$$y'(t) = k(t)y(\alpha t), y(0) \neq 0, \quad (136)$$

if  $k'(t) \geq 0$  and  $k(t) > k_0$  where  $k_0 > 0$  is a constant, then every solution has an infinity of negative zeros.

## 9 Boundary value problem of the second order linear pantograph equation and its applications

### 9.1 Boundary value problem of the second order linear pantograph equation

Consider the boundary value problem

$$y''(x) = \lambda y(\alpha^2 x), (0 < \alpha < 1), \quad (137)$$

with boundary condition

$$y(0) = y(1) = 0. \quad (138)$$

If  $\alpha = 1$ , then we have  $\lambda < 0$  for nontrivial solution. However, if  $0 < \alpha < 1$ , we need some efforts to proof this result.

**Lemma 9.1.** The function  $h(x) = E_\alpha(x) - E_\alpha(-x)$  has the unique real zero  $x = 0$ .

**Proof.** We can give a direct proof by the Taylor expansion of  $E_\alpha(x)$ . Here we give another proof. First, since  $h(-x) = E_\alpha(-x) - E_\alpha(x) = -h(x)$ , we know that if  $h(x_0) = 0$ , then  $h(-x_0) = 0$ . This means that if there is a positive zero, then there is a corresponding negative zero. Therefore, we only need to prove that  $h(x)$  has no positive zero. In fact,  $h(x)$  satisfies the following equation

$$h''(x) = \alpha h(\alpha^2 x), \quad (139)$$

with initial conditions  $h(0) = 0$  and  $h'(0) = 2$ . By the theorem 2.1, there exists the unique analytic solution. Thus, we can assume that the expansion of  $h(x)$  is

$$h(x) = \sum_{n=0}^{+\infty} b_n x^n, \quad (140)$$

where  $b'_n$ s are the undetermined coefficients. Substituting it into equation gives

$$b_{n+2} = \frac{b_n \alpha^{2n+1}}{(n+2)(n+1)}, n = 0, 1, 2, \dots \quad (141)$$

Since  $b_0 = 0$  and  $b_1 = 2$ , so  $b_{2n} = 0$  and  $b_{2n-1} > 0$  for  $n > 0$ , and then  $h(x)$  has no positive zero. The proof is completed.

Similarly, we have the following lemma.

**Lemma 9.2.** The function  $h(x) = E_\alpha(x) + E_\alpha(-x)$  has no any real zero.

By the above lemma 9.1, we have the following theorem.

**Theorem 9.1.** For the boundary value problem (137) and (138), if there is nontrivial solution, we must have  $\lambda < 0$ .

**Proof.** Assume that  $\lambda > 0$ . Then the general solution of equation (137) is give by

$$y(x) = c_1 E_\alpha(\sqrt{\frac{\lambda}{\alpha}}x) + c_2 E_\alpha(-\sqrt{\frac{\lambda}{\alpha}}x). \quad (142)$$

From the boundary conditions (141), we have

$$c_1 + c_2 = 0, \quad (143)$$

$$c_1 E_\alpha(\sqrt{\frac{\lambda}{\alpha}}) + c_2 E_\alpha(-\sqrt{\frac{\lambda}{\alpha}}) = 0. \quad (144)$$

By the lemma 9.1, the coefficients determinant is not zero, that is,

$$\begin{vmatrix} 1 & 1 \\ E_\alpha(\sqrt{\frac{\lambda}{\alpha}}) & E_\alpha(-\sqrt{\frac{\lambda}{\alpha}}) \end{vmatrix} \neq 0, \quad (145)$$

so  $c_1 = c_2 = 0$ , and then the solution  $y(x) \equiv 0$  is trivial. The proof is completed.

**Theorem 9.2.** For the boundary value problem (137) and (138), there exist an infinity of negative eigenvalues  $\lambda_n (n = 1, 2, \dots)$  which satisfy

$$\lambda_n = -\alpha \rho_n^2, \quad (146)$$

and every corresponding eigenfunction is given by

$$y_n(x) = S_\alpha(\rho_n x), \quad (147)$$

where  $\rho_n$  is the  $n$ -th positive zero of  $S_\alpha(x)$ .

**Proof.** By theorem 9.1, we must have  $\lambda < 0$ , then the general solution of the second order pantograph equation (137) is given by

$$y(x) = c_1 C_\alpha(\sqrt{-\frac{\lambda}{\alpha}}x) + c_2 S_\alpha(\sqrt{-\frac{\lambda}{\alpha}}x). \quad (148)$$

From  $y(0) = y(1) = 0$ , we have  $c_1 = 0$  and

$$c_2 S_\alpha(\sqrt{-\frac{\lambda}{\alpha}}) = 0. \quad (149)$$

Since  $c_2 \neq 0$ ,  $\lambda$  must satisfies

$$S_\alpha(\sqrt{-\frac{\lambda}{\alpha}}) = 0. \quad (150)$$

Denote all zeroes of  $S_\alpha(x)$  as  $\rho_0 = 0, \pm\rho_1, \pm\rho_2, \dots$ . So we have an infinity of  $\lambda$ 's as follows,

$$\lambda_n = -\alpha \rho_n^2, n = 1, 2, \dots. \quad (151)$$

Therefore, for each eigenvalue  $\lambda_n$ , an eigenfunction is given by

$$y_n(x) = S_\alpha(\rho_n x).$$

The proof is completed.

**Theorem 9.3.** For the symmetric boundary condition

$$y(-l) = y(l) = 0, \quad (152)$$

the eigenvalues of equation (137) must be negative and are given by

$$\lambda_{2n} = -\alpha \frac{\rho_n^2}{l^2}, \lambda_{2n-1} = -\alpha \frac{\eta_n^2}{l^2}, n = 1, 2, \dots, \quad (153)$$

and corresponding basic eigenfunctions are

$$y_{2n}(x) = S_\alpha\left(\frac{\rho_n}{l}x\right), \quad (154)$$

$$y_{2n+1}(x) = C_\alpha\left(\frac{\eta_n}{l}x\right), \quad (155)$$

where  $\rho_n$  and  $\eta_n$  are respectively the  $n$ -th zeros of  $S_\alpha(x)$  and  $C_\alpha(x)$ .

**Proof.** By lemmas 9.1 and 9.2, if there exists nontrivial solution, we must have  $\lambda < 0$ . In fact, if we suppose  $\lambda > 0$ , from the boundary conditions (152) and the general solution (142), we have

$$c_1 E_\alpha\left(\sqrt{\frac{\lambda}{\alpha}}l\right) + c_2 E_\alpha\left(-\sqrt{\frac{\lambda}{\alpha}}l\right) = 0 \quad (156)$$

$$c_1 E_\alpha\left(-\sqrt{\frac{\lambda}{\alpha}}l\right) + c_2 E_\alpha\left(\sqrt{\frac{\lambda}{\alpha}}l\right) = 0. \quad (157)$$

The coefficients determinant is

$$\begin{vmatrix} E_\alpha\left(\sqrt{\frac{\lambda}{\alpha}}l\right) & E_\alpha\left(-\sqrt{\frac{\lambda}{\alpha}}l\right) \\ E_\alpha\left(-\sqrt{\frac{\lambda}{\alpha}}l\right) & E_\alpha\left(\sqrt{\frac{\lambda}{\alpha}}l\right) \end{vmatrix} = E_\alpha^2\left(\sqrt{\frac{\lambda}{\alpha}}l\right) - E_\alpha^2\left(-\sqrt{\frac{\lambda}{\alpha}}l\right). \quad (158)$$

By the lemmas 9.1 and 9.2, the determinant is not equal to zero, so  $c_1 = c_2 = 0$ , and hence the solution  $y(x) \equiv 0$  is trivial. Therefore, we must have  $\lambda < 0$ , and the general solution is (148). By boundary conditions (152), we have

$$c_1 C_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right) + c_2 S_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right) = 0 \quad (159)$$

$$c_1 C_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right) - c_2 S_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right) = 0. \quad (160)$$

Since  $c_1$  and  $c_2$  can not be all zero, it follows that the determinant of the coefficients is zero, that is,

$$C_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right)S_\alpha\left(\sqrt{-\frac{\lambda}{\alpha}}l\right) = 0, \quad (161)$$

whose solutions are

$$\lambda_{2n} = -\alpha \frac{\rho_n^2}{l^2}, \lambda_{2n-1} = -\alpha \frac{\eta_n^2}{l^2}, n = 1, 2, \dots$$

The corresponding eigenfunctions can be taken as

$$y_{2n}(x) = S_\alpha\left(\frac{\rho_n}{l}x\right),$$

$$y_{2n-1}(x) = C_\alpha\left(\frac{\eta_n}{l}x\right),$$

for  $n = 1, 2, \dots$ . The proof is completed.

**Remark 9.1.** We must notice that these eigenfunctions are not orthogonal each other.

## 9.2 Formal solution of heat-like pantograph equation

Consider the following heat-like pantograph equation

$$u_t(\alpha^2 x, t) = u_{xx}(x, \beta t), \quad (162)$$

$$u(0, t) = u(1, t) = 0, \quad (163)$$

$$u(x, 0) = \phi(x). \quad (164)$$

If  $\alpha = \beta = 1$ , it will reduce to the usual heat equation.

We use the method of variables separation to solve the heat-like pantograph equation. Letting  $u(x, t) = X(x)T(t)$  and substituting it into the equation (162) yields two equations

$$T'(t) = \lambda T(\beta t), \quad (165)$$

and

$$X''(x) = \lambda X(\alpha^2 x), \quad (166)$$

with boundary condition

$$X(0) = X(1) = 0.$$

By the theorems 9.1 and 9.2, we have  $\lambda_n = -\alpha \rho_n^2$  for  $n = 1, 2, \dots$  and

$$X_n(x) = S_\alpha(\rho_n x), \quad (167)$$

and correspondingly

$$T_n(t) = E_\beta(-\alpha \rho_n^2 t). \quad (168)$$

Thus, formally, we have

$$u(x, t) = \sum_{n=1}^{+\infty} A_n E_\beta(-\alpha \rho_n^2 t) S_\alpha(\rho_n x), \quad (169)$$

which satisfies the heat-like pantograph equation and the boundary condition. We now use the initial condition to solve the coefficients  $A_n$ . In fact, taking  $t = 0$  in the above formula gives

$$\phi(x) = \sum_{n=1}^{+\infty} A_n S_\alpha(\rho_n x), \quad (170)$$

that is,  $\phi(x)$  can be expanded as a Fourier's series based on sine-like functions. Unfortunately,  $\{S_\alpha(\rho_n x)\}_{n=1}^{\infty}$  is not a system of orthogonal bases, and hence we can not direct compute the value of  $A_n$ . However, we can use the Gram-Schmidt's process to compute these coefficients. Denote  $f_n = S_\alpha(\rho_n x)$  for  $n = 1, 2, \dots$  and  $\{e_n\}_{n=1}^{+\infty}$  the corresponding orthogonal bases given by Gram-Schmidt's process. Then, for  $n = 2, 3, \dots$ , we have

$$\begin{aligned} e_1 &= f_1, \\ e_n &= f_n - \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 - \dots - \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1}, \end{aligned} \quad (171)$$

where inner product  $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ . Then  $e_n$  is the linear combination of  $f_1, \dots, f_n$ , that is,

$$e_n = \sum_{m=1}^n H_{nm} f_m, \quad (172)$$

where  $H_{nm}$  can be explicitly represented in terms of  $\frac{\langle f_i, e_j \rangle}{\langle e_j, e_j \rangle}$  for  $i, j = 1, \dots, n$ . Therefore, we get

$$\phi(x) = \sum_{m=1}^{+\infty} A_m f_m = \sum_{n=1}^{+\infty} B_n e_n = \sum_{n=1}^{+\infty} B_n \sum_{m=1}^n H_{nm} f_m = \sum_{m=1}^{+\infty} \sum_{n=m}^{+\infty} B_n H_{nm} f_m. \quad (173)$$

It follows that

$$A_m = \sum_{n=m}^{+\infty} B_n H_{nm}, \quad (174)$$

and

$$B_n = \frac{\langle \phi(x), e_n \rangle}{\langle e_n, e_n \rangle} = \frac{\sum_{k=1}^n \langle \phi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle}, \quad (175)$$

and hence, we have

$$A_m = \sum_{n=m}^{+\infty} \frac{H_{nm} \sum_{k=1}^n \langle \phi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle}. \quad (176)$$

Therefore, we give the formal solution of the heat-like pantograph equation

$$u(x, t) = \sum_{n=1}^{+\infty} \sum_{m=n}^{+\infty} \frac{H_{mn} \sum_{k=1}^m \langle \phi, f_k \rangle}{\sum_{k=1}^m H_{jk}^2 \langle f_k, f_k \rangle} E_\beta(-\alpha \rho_n^2 t) S_\alpha(\rho_n x). \quad (177)$$

**Remark 9.1.** This is only a formal solution, a strict treatment needs some refined knowledge about these special functions. I hope this problem and the same problem in next subsection can be solved in future.



### 9.3 Formal solution of wave-like pantograph equation

Consider the following wave-like pantograph equation

$$u_{tt}(\alpha^2 x, t) = u_{xx}(x, \beta^2 t), \quad (178)$$

$$u(0, t) = u(1, t) = 0, \quad (179)$$

$$u(x, 0) = \phi(x), \quad (180)$$

$$u_t(x, 0) = \varphi(x). \quad (181)$$

If  $\alpha = \beta = 1$ , we give the usual wave equation.

We use the method of variables separation to solve the wave-like pantograph equation. Letting  $u(x, t) = X(x)T(t)$  and substituting it into the equation (178) yields two equations

$$T''(t) = \lambda T(\beta^2 t), \quad (182)$$

and

$$X''(x) = \lambda X(\alpha^2 x), \quad (183)$$

with boundary condition

$$X(0) = X(1) = 0. \quad (184)$$

By the theorems 9.1 and 9.2, we must have  $\lambda < 0$  and denote  $\lambda_n = -\alpha\rho_n^2$  for  $n = 1, 2, \dots$ . Moreover, by boundary conditions, we have

$$X_n(x) = S_\alpha(\rho_n x), \quad (185)$$

and correspondingly

$$T_n(t) = A_n C_\beta(-\rho_n t) + B_n S_\beta(-\rho_n t). \quad (186)$$

Thus, formally, we have

$$u(x, t) = \sum_{n=1}^{+\infty} \{A_n C_\beta(-\rho_n t) + B_n S_\beta(-\rho_n t)\} S_\alpha(\rho_n x), \quad (187)$$

which satisfies the wave-like pantograph equation and the boundary condition. We now use the initial conditions to solve the coefficients  $A_n$  and  $B_n$ . In fact, by the initial conditions, we have

$$\phi(x) = \sum_{n=1}^{+\infty} A_n S_\alpha(\rho_n x), \quad (188)$$

and

$$\varphi(x) = - \sum_{n=1}^{+\infty} B_n \rho_n S_\alpha(\rho_n x), \quad (189)$$

that is,  $\phi(x)$  and  $\varphi(x)$  can be expanded as a Fourier's series based on sine-like functions. By the similar method with the heat-like pantograph equation, we have

$$A_m = \sum_{n=m}^{+\infty} \frac{H_{nm} \sum_{k=1}^n \langle \phi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle}, \quad (190)$$

and

$$B_m = -\frac{1}{\rho_m^2} \sum_{n=m}^{+\infty} \frac{H_{nm} \sum_{k=1}^n \langle \varphi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle}, \quad (191)$$

where  $f_n = S_\alpha(\rho_n x)$ . Therefore, the formal solution of the wave-like pantograph equation is given by

$$u(x, t) = \sum_{m=1}^{+\infty} \left\{ \sum_{n=m}^{+\infty} \frac{H_{nm} \sum_{k=1}^n \langle \phi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle} C_\beta(-\rho_m t) \right. \quad (192)$$

$$\left. - \frac{1}{\rho_m^2} \sum_{n=m}^{+\infty} \frac{H_{nm} \sum_{k=1}^n \langle \varphi, f_k \rangle}{\sum_{k=1}^n H_{nk}^2 \langle f_k, f_k \rangle} S_\beta(-\rho_m t) \right\} S_\alpha(\rho_m x). \quad (193)$$

If we consider the wave-like pantograph equation on the infinite interval  $(-\infty, +\infty)$ ,

$$u_{tt}(\alpha^2 x, t) = u_{xx}(x, \beta^2 t), \quad (194)$$

$$u(-\infty, t) = u(+\infty, t) = 0, \quad (195)$$

$$u(x, 0) = \phi(x), \quad (196)$$

$$u_t(x, 0) = \varphi(x), \quad (197)$$

we will give the formal solution

$$\begin{aligned} u(x, t) = & \int_{-\infty}^{+\infty} S_\beta\left(\frac{y}{\sqrt{\beta}}t\right) \{A_1(y)S_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right) + B_1(y)C_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right)\} dy \\ & + \int_{-\infty}^{+\infty} C_\beta\left(\frac{y}{\sqrt{\beta}}t\right) \{A_2(y)S_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right) + B_2(y)C_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right)\} dy, \end{aligned} \quad (198)$$

where  $A_k(y)$  and  $B_k(y)$  ( $k = 1, 2$ ) satisfy

$$\phi(x) = \int_{-\infty}^{+\infty} \{A_1(y)S_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right) + B_1(y)C_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right)\} dy, \quad (199)$$

and

$$\varphi(x) = \int_{-\infty}^{+\infty} \frac{y}{\sqrt{\beta}} \{A_2(y)S_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right) + B_2(y)C_\alpha\left(\frac{y}{\sqrt{\alpha}}x\right)\} dy. \quad (200)$$

We can introduce a Fourier-like transformation

$$f(x) = \int_{-\infty}^{+\infty} F(y)E_\alpha(iyx) dy, \quad (201)$$

which is similar to the usual Fourier transformation, but we don't have a simple inverse transformation formula. A possible method is to consider the interval  $(-l, l)$  and take a limitation process of  $l \rightarrow +\infty$ . However, the corresponding formula will be very complicated.

**Remark 9.2.** We can also define a Laplace-like transformation

$$L(f)(p) = \int_0^{+\infty} E_\alpha(-px)f(x)dx, \quad (202)$$

but I do not know how to give its inverse transformation. I leave these as the open problems.

## 10 Conclusion

The multiplication delay functional differential equations namely also pantograph equations provide a mathematical tool to describe and simulate the physical phenomenons with the time-dependent memory. At the same time, the theory of this kind of equations have also independent interest for mathematicians. By using three special functions, namely exponent-like function  $E_\alpha(x)$ , cosine-like function  $C_\alpha(x)$  and sine-like function  $S_\alpha(x)$ , we obtain explicitly the structures of the solutions of the linear pantograph equations. Furthermore, we obtain the results on the existence, uniqueness and non-uniqueness for the initial value problems at a general point. These results show the usages and meanings of these three special functions. These results also show that there are many similar characters between the usual differential equations theory and the pantograph equations, but there also exist many serious differences.

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